

T H E
G E O M E T R I C I A N :
C O N T A I N I N G
E S S A Y S
O N
P L A N E G E O M E T R Y ,
A N D
T R I G O N O M E T R Y :

With their Application to the SOLUTIONS of a
VARIETY of PROBLEMS, which are of great Use
in *measuring Heights and Distances of Places, sur-
veying of Counties, Sea-Coasts, &c.*

By BENJAMIN DONN,
Teacher of the *Mathematics*, and Lecturer in *experimental
Philosophy*,
Author of the *Essays on Arithmetic, Book-keeping, the
British Mariner's Assistant, &c.*

The SECOND EDITION.

*Quicquid in astronomicis, geographicis, vel nauticis, efficiendum,
geometriæ et trigonometriæ attribuendum est.*

L O N D O N :
Printed for J. JOHNSON, in *St. Paul's Church-Yard*.
M.DCC.LXXVIII.



T O T H E
T U T O R S I N T H E U N I V E R S I T I E S ,
M A S T E R S O F A C A D E M I E S ,
A N D O T H E R
T E A C H E R S O F G E O M E T R Y ,
I N G R E A T - B R I T A I N .

G E N T L E M E N ,

IT is probable that many of you, as well as myself, have found it frequently difficult to initiate young Gentlemen into a necessary Acquaintance with Geometry by the Elements of *Euclid*, and therefore have wished for a more easy Introduction to that valuable Science, which, at the same time as it facilitated the Attainment of Geometry, might not depart (as many do) too much from the geometrical Spirit of the Ancients, which is so necessary for acquiring a Habit of reasoning with Propriety and Judgement.

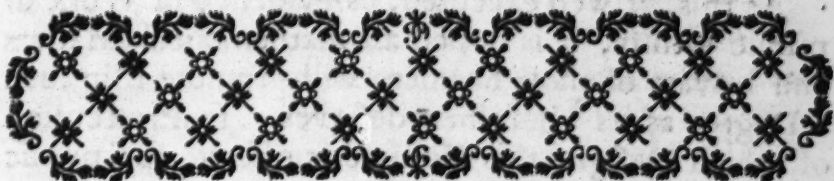
Such a Treatise, with its Application to Trigonometry, I have now attempted, and beg Leave to offer to the Public, under your Protection; and, if it should be so far approved of, by you, as to be put into the Hands of your Pupils, I flatter myself you will find the Science will be acquired by them in much less Time than is usual, and with greater Ease to yourselves. If this meets with your Encouragement, it will induce me to pursue my original Design, of presenting the Public with other Essays, for rendering the several Branches of mathematical Literature, both ancient and modern, more pleasant in the Study, and more easily to be attained, than by any other Course, hitherto published, in our Language.

I am, Gentlemen,

Kingston, near Taunton,
July 5, 1775.

Your humble Servant,

B. D O N N .



THE PREFACE.

IN the general Preface to the *Essays on Arithmetic*, published in the Year 1758, the Public were acquainted with our Design of presenting them with a new Course of mathematical Learning, a Thing generally allowed to be much wanted: * For the great and numerous Improvements, which have been made since any Course has been published in the *English* Language, make a new Course absolutely necessary. For, as those Sciences, with their Improvements, are dispersed in a Multitude of Authors, the young Student knows not how to proceed; and, even to a Master, it is no easy Task to direct: Whereas, if these Sciences are brought into a regular Course, as they depend on each other, the Student will learn with more Ease, Pleasure, and Dispatch.

As

* “ Mathematical Learning, during the last and present Centuries, has made a most surprizing Progress; and Truth, assisted by the uncontroverted Principles of this Science, has banished hypothetical Chicanery from the Regions of Philosophy. It is, therefore, no Wonder that a great Variety of Authors, desirous of extending so valuable a Branch of Science, should have written on every Part of mathematical Learning. But still a Course of Mathematics and Natural-Philosophy, tracing the Science from its first Principles, and exhibiting the Demonstrations on which each Rule or Problem is founded, is still wanting; there being none, in our own Language, that can, with any Show of Justice, be called a Course of Mathematics and Natural-Philosophy, according to the modern Improvements, and properly adapted to Learners. This Defect Mr. *Donn* has undertaken to supply.”

MONTHLY REVIEW for JULY, 1758.

P R E F A C E.

As this, if well executed, is evidently a Work of public Utility, it is hoped all mathematical Masters and Lovers of those Sciences will promote it in such a Degree as its Usefulness deserves. In Prosecution of which Design, this Volume is offered for public Acceptance, containing Essays on *Logarithms*, *Plane Geometry* and *Trigonometry*.

It may, perhaps, be objected, by some, that, since the first Edition of this Volume was printed, several Volumes of a Course have been published, by an ingenious Gentleman, and that therefore a new Course of the Mathematics is not now necessary. But I beg Leave to observe, that the Masters of Academies, and other Teachers, must be sensible that *that* Course is not drawn up in a Manner proper to introduce Learners to those difficult Sciences, as, in several of the Volumes, the difficult Parts of the Subjects are treated of before the more easy, &c. Of this the learned Author was so sensible, that, on my acquainting him (when I had the Pleasure of seeing him in *London*) that I obtained my first Knowledge of Fluxions from his excellent Treatise, without the Assistance of a Master, when I was about 18 Years of Age; he answered, with a Kind of Surprise, "From *my* Fluxions! My Treatise was intended for *Masters*, not for *Learners*." I have not made this Remark either out of any high Opinion I entertain of my own Abilities or to depreciate his Works, but only to give his own Idea of his Publications. I esteem him as an excellent Mathematician, and shall therefore only observe farther, that, though it would perhaps be thought Presumption in me to promise to treat of the same Subjects in a more masterly Manner, yet, I may venture to say, I hope to give them on a much more enlarged Plan, with respect to practical Mathematics, and more easy to be understood by young Mathematicians.

Notwithstanding

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Notwithstanding it is intended to treat of the several Sciences in successive Order, as they ought to be studied, to form a regular Course, for the Sake of such Readers as are desirous of acquiring a general Knowledge therein; yet, at the same Time, on Account of Others, who may only be inclined to study such Parts as may be more particularly useful to them, we shall endeavour to make each Subject distinct and entire by itself, so that every Person may have it at his own Choice to read what Subjects he pleases.

As this Work is written as a Course, the Reader of any Essay is supposed to know no more of the Mathematics than is contained in the preceding Essays, and therefore must not expect any Theorems, &c. which cannot be demonstrated by the foregoing; so that if he happens to find any valuable Things omitted, in any Essay of ours, which he has seen in any other Work, we would not have him think we have forgot to insert it, but that we refer it to a more proper Place, in a future Essay. For Instance, in Logarithms, we have omitted the Investigation of Series, because it depends on the higher Parts of the Mathematics, and must therefore be omitted until those Parts are explained.

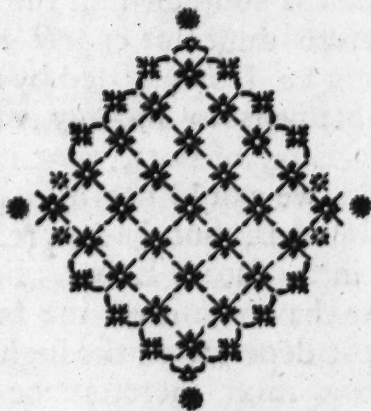
Essays on Circulating-Decimals and Book-Keeping were prefixed to the first Edition of our *Geometrician*, and published under the Title of *The Accountant and Geometrician*; but, as they have no Connection with Geometry, it was thought best to omit them in this Edition: And, as the Essays on Book-Keeping have been enlarged in Number, by the Addition of *The Shopkeeper's, Steward's, and Factor's, Companion*, it was thought best to publish them in a Volume by themselves, entitled, *The Accountant*; and, instead of them, in this Volume, to give the *Elements* † of *Plane Trigonometry*,

† To oblige the Purchasers of the first Edition of our *Geometry*, we have printed a larger Number of the *Trigonometry*, that they may not be put to the Expence of purchasing the whole Volume a second Time; but it will only be sold separate to those who have already the first Edition.

P R E F A C E.

Trigonometry, with their Application to *Altimetry* and *Longimetry*. For a more circumstantial Account of the Contents of this Volume, we beg Leave to refer the Reader to the Titles and Prefaces to the several *Essays*.

B. D O N N.



Just published,

(From the AUTHOR'S MANUSCRIPT,)

A S E R M O N,

Preached at the Funeral of Mr. ABRAHAM DONN,

By the late pious and ingenious

JAMES HERVEY, M. A.

Rector of *Weston-Favel*, in *Northamptonshire*, and Author of the
MEDITATIONS and CONTEMPLATIONS.

Printed for B. LAW, in AVE-MARIA-LANE,

AN
E S S A Y
ON THE
NATURE, INVESTIGATION, and
APPLICATION,
OF
LOGARITHMS.

By BENJAMIN DONN.

*Si numeris in ratione geometrica progredientibus
subscribantur totidem alii æquidifferentes; di-
cuntur hi illorum Logarithmi.*

WOLFIUS, Elem. Arith.

ESSAYS

ON THE

NATURE, INVESTIGATION, AND
APPLICATION

OF

LOGARITHMS.

By BENJAMIN DOWNS.

Si numerus in ratione geometrica progressionem
habebit, tunc et logarithmus eius
erit in ratione arithmetica.
Vossius, Elem. Arith.

P R E F A C E.

THE Invention of *Logarithms* is universally allowed to be one of the most useful Discoveries in the Art of Numbers: For, as Dr. Keil justly observes, “ It is by their Means “ that Numbers almost infinite, and such as “ are otherwise impracticable, are managed “ with ease and expedition. By their Assistance “ the Mariner steers his Vessel, the Geometri- “ cian investigates the Nature of the higher “ Curves, the Astronomer determines the Places of the Stars, and the Philosopher accounts “ for other Phenomena of Nature; and lastly, “ the Usurer computes the Interest of his Money.”

The Honour of the Invention of Logarithms is sufficiently secured to Lord *John Neper*, Baron of *Merchiston* in *Scotland*, by the Testimonies of all who have treated on the Subject, as well Foreigners, as Inhabitants of *Britain*.

In 1614, Lord *Neper* published at *Edinburgh* a Canon of Logarithms, in which 10 is taken for the Logarithm of Radius (or 100000000000,) and the Logarithms of the Sines are affirmative.

This Book being in *Latin*, was translated into *English* by Mr. *Edward Wright*, and after his Decease published by his Son *Samuel*, in the Year 1716.

The great Usefulness of Logarithms was immediately perceived by the learned Mathematicians of the Age; and amongst them Mr.

P R E F A C E.

Briggs (then Professor of Geometry at *Gresham College, London,*) was so charmed with the noble Invention, that he went to *Scotland* on Purpose to consult with the Inventor, and assist him in perfecting his Design: The Result of their Consultation was, that the Inventor and *Mr. Briggs* agreed to change the Form of the Logarithms into a more compendious one; in which the Logarithm of 1 was to be 0; and 1000000, &c. the Logarithm of Radius.

In the Year 1619, the noble Inventor being dead, his Son Lord *Robert* published the Construction of the *admirable Canon*; in the Preface to which he takes Occasion to mention the great Friendship his Father had for *Mr. Briggs*; on whom now the whole Labour of calculating Tables entirely devolved. This did not discourage *Mr. Briggs*, for applying himself to the Work with a Degree of Industry that surprized the whole World, he published at *London*, in 1624, his *Arithmetica Logarithmica*, agreeable to the new Form, for the first 20 Chiliads of Logarithms, (or from 1 to 20000,) and for 11 Chiliads more, *viz.* from 90000 to 101000, calculated to 14 Places of Figures. He also gave Directions for supplying the intermediate Chiliads.

In the Year 1627, *Adrian Vlac* (or *Fack*) published a Canon to 10 Places of Figures, in which the intermediate Chiliads were filled up, according to *Mr. Briggs's* Direction.

In 1633, *Vlac* published *Brigg's* Logarithms of Sines and Tangents to every 10 Seconds, and of Numbers from 1 to 20000.

In

PREFACE

In the Year 1658, Dr. John Newton published at London *Trigonometria Britannica*, in which are Tables of Logarithms of Numbers from 1 to 100000, and also the Logarithms of Sines and Tangents to $\frac{1}{100}$ Parts of every Degree.

The Editions of the Tables of Logarithms now most esteemed are Sberwin's, containing the Logarithms taken from Dr. John Newton; and Gardener's, containing the Logarithms of Numbers taken also from Dr. John Newton, and the Logarithms of Sines and Tangents to every 10 Seconds, taken from Vlac's *Canon Magnus*.

We must not however forget Mr. Dodson's *Antilogarithmic Canon*, published in the Year 1742; being a Table of Numbers, consisting of 11 Places of Figures, corresponding to all Logarithms under 100000; which indeed is a laborious and ingenious Work.

Other Authors who have laboured in completing Tables, &c. are, in the Year 1619, Speidell and Urfinus; in 1620, Gunter; in 1624, Wingate; in 1625, Urfinus again, and Kepler; in 1626, Henrion; in 1627, Speidell, and Kepler again in his *Rudolphine Tables*; also sometime before this, Gellibrand, as in Vlac's *Trigonometria Artificialis*, published this Year, the explanatory Part is an Abridgement of Gellibrand's Trigonometry. In the Year 1631, Norwood; in 1632, Cavalerius; in 1633, Rowe; in 1634, Frobenius; in 1670, Caramuel: Since which Time, Dr. Gregory, Mercator, Sir Isaac Newton, Dr. Halley, William Jones, Esq. and a Number of other learned Men,

P R E F A C E.

Men, have published infinite converging Series, for computing Logarithms more expeditiously; some of which we may have Occasion to explain hereafter.

It would be endless to give a List of all the Authors who have touched on Logarithms; because, almost all who have treated on Trigonometry, on Navigation, &c. have given compendious Tables of Logarithms, &c.

By this compendious ESSAY, it is hoped the Learner may attain as clear Ideas of the Nature and Application of Logarithms, as he could by much larger Treatises.

Proper Tables of Logarithms will be given in the Essay on *Trigonometry*, or *Navigation*, as they will there be of more immediate Use, &c. as the Tables of Sines, Tangents, &c. could not with any Propriety be given in this Place.

Bideford, February 7,

1759.

B. DONN.



A

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A N
E S S A Y
O N
Logarithmical Arithmetick.

C H A P. I.

*Of DEFINITIONS, and the Nature and Investigation
of LOGARITHMS.*

I. **L**OGARITHMS (*λόγος* and *ἀριθμός*) are Numbers so contrived and adapted to other Numbers, that by their Addition, or Subtraction, the Product, or Quotient, of the Numbers to which they are adapted, may be found.

This Definition expresses the principal Design and Use of those artificial Numbers, which Mathematicians distinguish by the Name of *Logarithms*. But here, (as it frequently happens in other Sciences,) a clear, accurate, etymological Sense, cannot be understood, till we have made a particular Enquiry into the Principles of the Science itself.

2. To proceed then, let us consider the Nature of a geometrical Progression, with the Indices set over the respective Terms.

B

For

NATURE OF LOGARITHMS.

For Example :

Indices, 0. 1. 2. 3. 4. 5. 6. &c.

Terms, 1. 2. 4. 8. 16. 32. 64. &c.

Here it is evident, that if we add any two Indices together, their Sum will be the Index of that Number, which is equal to the Product of the Numbers whose Indices are added together : Thus, *e. g.* The Indices 2 and 3 added together are $= 5$; the Numbers answering to the Indices 2 and 3, are 4 and 8 ; and $4 \times 8 = 32 =$ the Number answering to the Index 5.

It also follows from the Nature of such Progressions, that if we subtract one Index from another, their Difference will be the Index of that Number or Term, which is equal to the Quotient of the two Numbers or Terms, corresponding to the Indices whose Differences we found : *e. g.* The Indices $6 - 4 = 2$; the Numbers corresponding to these Indices are 64 and 16, and $64 \div 16 = 4 =$ the Number corresponding to the Index 2.

3. Hence it follows, that (by Art. 1.) the Indices of a Series of Numbers in geometrical Progression, are Logarithms of the Terms in that Progression.

4. Hence, by the Nature of geometrical Progressions, it follows, that, if the Logarithm of any Number be multiplied by the Index of the Power, the Product will be equal to the Logarithm of the Root when involved to the Height denoted by the Index. Thus, for Instance : The Index or Logarithm of 4 in the above Series 2, being multiplied by 3 $= 6 =$ the Index or Logarithm of 64 ; and $64 = 4$ cubed $= 4 \times 4 \times 4$.

5. From hence also it follows, that the Index or Logarithm of any Number being divided by 2, the Quotient will be the Index or Logarithm of the Square Root ; but if divided by 3, the Index or Logarithm of the cube Root ; if divided by 4, the Quotient will be the Index or Logarithm of the 4th Power, &c.

Example.



Example. The Logarithm of 64 in the above Series is 6, which being divided by 3, the Quotient is 2, the Index or Logarithm of 4, which is the cube Root of 64; for $4 \times 4 \times 4 = 64$.

6. It follows, from the Nature of geometrical Progressions, that the Ratio of the first, and any other Term, is composed of so many equal Ratios, as are expressed by the Index of that other Term: *e. g.* In the above Series, the Ratio of the first and 7th Term is made up of 6 equal Ratios, for the Ratio of the 1st to the 2d Term is as 1 to 2, and of the 2d to the 3d as 1 to 2, and of the 3d to the 4th as 1 to 2, of the 4th to the 5th as 1 to 2, of the 5th to the 6th as 1 to 2, and of the 6th to the 7th as 1 to 2; $\therefore$ the Ratio of the 1st to the 7th is composed of $1 \times 1 \times 1 \times 1 \times 1 \times 1$ to $2 \times 2 \times 2 \times 2 \times 2 \times 2$; *i. e.* as 1 to 64. Hence plainly appears the Propriety of the Term *Logarithms*, it signifying, according to its Etymology, a *Number of Ratios*.

7. Logarithms may be of various Kinds; for instead of the Series 1, 2, 4, &c. above, we may substitute any other; thus *e. g.* if we write the geometrical Series 1, 10, 100, &c. instead of 1, 2, 4, &c. it will stand thus:

Indices or Logarithms 0. 1. 2. 3. 4. &c.
Terms or natural N^{os}. 1. 10. 100. 1000. 10000, &c.

This is the Form of *Brigs's* Logarithms; which are those we shall now make it our Business to discourse on.

The greatest Difficulty now remaining is to explain the Method made Use of by Mr. *Brigs*, for finding the Logarithms of the intermediate Terms, because the natural Numbers, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c. are not in a geometrical Progression.

8. Before we proceed any farther, it may be proper to observe, that, if we suppose a geometrical Series beginning from Unity, whose common Multiplier is $1 + m$, the Series will be 1, $1 + m$,

$1 + m^2$, $1 + m^3$, $1 + m^4$, &c. Now if m be supposed

INVESTIGATION OF LOGARITHMS.

posed indefinitely little, it is evident that some of the Terms will approach indefinitely near to the natural Numbers 2, 3, 4, 5, &c. and, consequently, it is possible to find the natural Number answering to any Logarithm, very nearly, though not exactly true.

9. The Method of doing which may be thus explained. — By the Definition of Logarithms, if x and y represent any two Numbers, the Logarithm of

$x \times y$ = the Logarithm of x + the Logarithm of y ;
and therefore, (by Art. 5.) the Logarithm of

$$\sqrt{x \times y} = \frac{\text{the Log. } x + \text{Log. } y}{2} : \text{Hence, the } \textit{arith-}$$

metical Mean of any two Logarithms is equal to the Logarithm of the *geometrical Mean* of the two natural Numbers; consequently, the Logarithm of any intermediate Number may be found, by finding the geometrical Mean between the two given Numbers, (whose Logarithms are given,) and the Logarithm answering thereto, (*viz.* the arithmetical Mean of the two given Logarithms :) Then, the geometrical Mean between the geometrical Mean just found and the nearest Extreme, and the Logarithm answering thereto. After this Manner we proceed, till we find a geometrical Mean so very nearly equal to the Number whose Logarithm we want, as we shall think sufficiently near the Truth; then the Logarithm of that geometrical Mean is sufficiently near the Logarithm required.

10. To illustrate this, let it be required to find the Logarithm of 9. Here we have the Logarithms of 1 and 10 given.

* Half the Sum of any two Numbers is called an *arithmetical Mean*; and the square Root of the Product of two Numbers is called the *geometrical Mean*.

INVESTIGATION OF LOGARITHMS.

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(1.) The Log. of 1 = 0
 Log. of 10 = 1 And $\sqrt{1 \times 10}$

$$\begin{array}{r} \text{Sum} = 1 \\ \hline \frac{1}{2} \text{ Arithmetical Mean} = 0.5 \end{array}$$

$$\begin{array}{r} = \sqrt{10} = 3.1622 \\ 777, \therefore \text{the Log.} \\ \text{of } 3.1622777 \text{ is} = \\ 0.5. \end{array}$$

(2.) Hence we have the Logarithms of 3.1622777 and 10 given.

Log. of 10 = 1.
 Log. of 3.1622777 = 0.5

$$\begin{array}{r} \text{Hence the Log. of} \\ \sqrt{10 \times 3.1622777}, \\ \text{viz. of } 5.6234132 \text{ is} \\ \hline 1.5 \\ \frac{1}{2} \text{ } = 0.75. \end{array}$$

(3.) We have now the Logarithms of 10 and 5.6234132 given.

Numbers $\left\{ \begin{array}{l} 10. - - - - \text{its Log.} = 1. \\ 5.6234132 - \text{its Log.} = 0.75 \end{array} \right.$

$$\begin{array}{r} \hline 1.75 \\ \frac{1}{2} \text{ } = 0.875 \end{array}$$

$\sqrt{10 \times 5.6234132} = 7.4989421$ its Log. = 0.875

(4.) Here we have the Log. of 10 and 7.4989421 given.

Numbers $\left\{ \begin{array}{l} 10. \\ 7.4989421 \end{array} \right\}$ Their Logarithms $\left\{ \begin{array}{l} 1. \\ 0.875 \end{array} \right.$

$$\begin{array}{r} \hline 1.875 \\ \frac{1}{2} \text{ } = 0.9375 \end{array}$$

$\sqrt{10 \times 7.4989421} = 8.6596431$ its Log. = 0.9375

(5.) Hence we have the Logarithms of 10 and 8.6596431 given.

Numbers $\left\{ \begin{array}{l} 10. \\ 8.6596431 \end{array} \right\}$ Their Logarithms $\left\{ \begin{array}{l} 1. \\ 0.9375 \end{array} \right.$

$$\begin{array}{r} \hline 1.9375 \\ \frac{1}{2} \text{ } = 0.96875 \end{array}$$

$\sqrt{10 \times 8.6596431} = 9.3057204$ its Log. = 0.96875
 (6.)

OF LOGARITHMS OF FRACTIONS.

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consider the great Labour and Patience necessary to the Accomplishment of so great a Work, how much are we beholden to Lord *Neper* and Mr. *Brigs*? Certainly their Names will, and ought to be perpetuated from one Age to another, even to the End of the World!

There have lately been invented much easier Methods of constructing Logarithms than the above; but as they depend on Fluxions, Conic Sections, &c. they cannot be understood in this Place. And it is sufficient to have shewn the Method by which the Tables were made, as we have now no Occasion to construct new ones.

15. Having explained the Nature of making a Table of Logarithms of Numbers greater than Unity, the next Thing to be done is to shew how the Logarithms of fractional Numbers may be found. In Order to which let it be observed, that, as we have hitherto supposed a geometrical Series to increase from an Unit on the right Hand, we may now suppose it to decrease from an Unit towards the left Hand, as here annexed.

Log. —3.	—2.	—1.	0.	+1.	+2.	+3, &c.
C.			A.			B.
Numbers	$\frac{1}{1000}$	$\frac{1}{100}$	$\frac{1}{10}$	1.	10.	100. 1000, &c.

Here the Series from the Point A, or Unity, increases towards B, and decreases towards C.

It is evident by a bare Inspection, that, if the Numbers are both Fractions, the Sum of their Logarithms is equal to the Logarithm of their Product. But if one Number is on the Right-hand of the Point A, and the other on the Left of it; that is, if one Number is greater than an Unit, and the other a Fraction, then the Difference of their Logarithms will be equal to the Logarithm of their Product; and the Logarithm of the Product will be on the same Side of the Point A, as is the greater of the two Logarithms.

OF LOGARITHMS OF FRACTIONS.

16. In Order to make a proper Distinction, we may denote the Logarithm of any Number greater than an Unit by the Sign +, for they may be reckoned affirmative; and then the fractional Quantities must be considered as negative.

17. It is also evident from Inspection of the above Series, that if a Number, having an affirmative Logarithm, is divided by another Number whose Logarithm is a greater affirmative Number, the Difference of their Logarithms taken negatively will be the Logarithm of their Quotient. Thus, *e. g.* the Logarithm of 100 = 2, and the Logarithm of 1000 = 3; their Difference taken negatively is -1 = the Logarithm of $\frac{1}{10}$, the Quotient of $100 \div 1000$.

— This might have been deduced from the Nature of Fractions. For the Numerator of any Fraction may be esteemed as a Dividend, and the Denominator as a Divisor: Therefore, by the Nature of Logarithms, (Art. 1.) the Logarithm of the Numerator *minus* the Logarithm of the Denominator is equal to the Logarithm of the Fraction: But since the Denominator of a proper Fraction is always greater than the Numerator, its Logarithm will be greater than that of the Numerator, and consequently it will be impossible to take the Logarithm of the Denominator from that of the Numerator; therefore, we are obliged to take the Logarithm of the Numerator from that of the Denominator, and express the Difference negatively, for the Logarithm of the Fraction.

18. The Logarithm of 1, 10, 100, 1000, &c. being 0, 1, 2, 3, &c. respectively, it is manifest, that the Logarithms of the intermediate Terms may be considered as a mixed Number, made up of an Integer and decimal Fraction; and that the integral Part, called by some Writers the *Index*, (*Lat.*) by Others the *Exponent*, (from *expono*, *Lat.*) and by Others the *Characteristick*, (from *Character*, *Lat.*) is one less than the Number which expresses how many Figures the natural Number consists of; *e. g.* if the natural

OF LOGARITHMS OF FRACTIONS.

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natural Number has 4 Places, the Characteristick of the Logarithm will be 3; if of 3 Places, 2; if of 2 Places, 1, &c.

19. Division being performed by Subtraction of the Logarithms, and the Logarithm of 10 being 1, it follows, that the Logarithm of $\frac{1}{10}$ of any Number, must be one less than the Logarithm of that Number. Thus, e. g. the Logarithm of 7851 being + 3.894925,

The Log. of	$\frac{7851}{10} = 785.1$	} is {	$+ 2.894925$	Affirmative Logarithms.
	$\frac{785.1}{10} = 78.51$		$+ 1.894925$	
	$\frac{78.51}{10} = 7.851$		$+ 0.894925$	
	$\frac{7.851}{10} = .7851$		$- 0.105075$	Negative Logarithms.
	$\frac{.7851}{10} = .07851$		$- 1.105075$	
	$\frac{.07851}{10} = .007851$		$- 2.105075$	
	&c.			

The above affirmative Logarithms were found by successively subtracting the Logarithm of 10, viz. + 1 from the preceding Logarithm; then, to find the first negative Logarithm, + 1 (the Log. of 10) was to be taken from the Logarithm + 0.894925; but as the Log. to be subtracted is greater than that from which it was to be taken, we are obliged to subtract the lesser 0.894925 from the greater 1, and express their Difference negatively, for the required Logarithm, (as directed in Art. 17,) which gives - 0.105075. Then to find the other negative Logarithms, the Logarithm + 1 being to be subtracted from a negative Logarithm, we have only to add 1 successively, and express the Sum negatively; for by Art. 46, *Essay on Arithmetick*, to subtract an Affirmative is the same as to add a Negative.

OF BINOMIAL LOGARITHMS.

20. The Logarithms of Fractions may be expressed in such a Manner, that the fractional Part may be affirmative, and the Characteristick negative. Logarithms thus expressed are called *binomial Logarithms* (from *binomius*, *Lat.*) and are better adapted for Practice than the negative ones.

21. To find the binomial Logarithm of a proper Fraction,

* From the Logarithm of the Numerator subtract that of the Denominator; remembering, that when we come to the Characteristicks, the Characteristick of the Numerator is to be taken from that of the Denominator; (after having added the one that was borrowed, if any, to the Characteristick of the Denominator.)

* That the binomial Logarithms found by this Rule, are, in Effect, the same as the negative Logarithms afore described, may

be thus shewn. Let $\frac{n}{d}$ be any proper Fraction; then, putting c

for the integral, and r for the fractional Part, of the Logarithm of n ; we have $c + r = \text{Logarithm of } n$; and putting t for the Characteristick, and $a = \text{the decimal Part of the Logarithm of } d$, then $t + a = \text{the Logarithm of } d$. Hence, by Art. 17,

$\overline{t+a} - \overline{c+r} = \overline{t+a-c-r}$ taken negatively, (that is, changing the Signs,) viz. $-t-a+c+r = \text{the Logarithm of } \frac{n}{d}$.

But for the binomial Logarithm we have two Cases. 1st, When r is greater than a , then $t - c$ taken negatively is $-t + c$, $\therefore$ the binomial Logarithm, according to this Rule, is $-t + c + r - a$, which may stand thus, $-t - a + c + r$, which is the same as the above Expression for the negative Logarithm. 2dly,

When r is less than a . First $c + r = \overline{c-1} + \overline{1+r} = \text{the Log. of } n$; now subtracting a from $1 + r$, there remains $1 + r - a$; and taking $c - 1$ from t , (according to this Rule,) or, which is the same Thing, c from $t + 1$, there remains $t + 1 - c$, which taken negatively is $-t - 1 + c$ for the Characteristick; $\therefore -t - 1 + c + (\text{the decimal Part of the Logarithm}) 1 + r - a = \text{the binomial Logarithm} = -t + c + r - a = -t - a + c + r$, the same Expression as that above, for the negative Logarithm.
Q. E. D.

Example.

Example. The binomial Logarithm of $\frac{7.851}{10}$ is
 $= -1 + .894925$.

For the Logarithm of $7.851 = 0.894925$ by Art. 19.

The Logarithm of $10 = 1.000000$

The Log. of $\frac{7.851}{10}$ by this Rule $= -1. + 894925$.

But this Logarithm is more commodiously wrote
 thus, $\overline{1}.894925$, the Dash over the Characteristick
 shewing it is a negative Quantity, and the remaining
 Part affirmative.

Hence the Log. of $\frac{7.851}{10} = .7851$ is $= \overline{1}.894925$.

of $\frac{.7851}{10} = .07851$ is $= \overline{2}.894925$.

of $\frac{.07851}{10} = .007851$ is $= \overline{3}.894925$.

Ec.

Ec.

Hence it is evident, that the decimal Part of the
 binomial Logarithm of a decimal Fraction, is the
 same as that of an integral Number of the same Fi-
 gures; and it is also manifest, that if the significant
 Figure of the decimal Number is in the first Place,
 the Characteristick is -1 ; if the significant Figure
 be in the second Place, the integral Part of the Loga-
 rithm is -2 , *Ec.* And this is the Kind (of Loga-
 rithm) we shall make Use of in the remaining Part
 of this ESSAY, to express fractional Quantities.

22. In taking out of a Table the Logarithm of
 any Integer not exceeding 10000, we have the frac-
 tional Part by Inspection; to which we prefix for a
 Characteristick 3, if the Number consists of 4 Fi-
 gures; 2, if of 3 Figures, *Ec.* that is, the Cha-
 racteristick must be one less than the Number of Pla-
 ces: But if the Number, whose Logarithm is re-
 quired, be above 10000; then, find the Logarithm
 of the two nearest Numbers to it that can be found

OF BINOMIAL LOGARITHMS.

in the Table, and say, As their Difference : the Difference of their Logarithms :: the Difference between the nearest Number and that whose Logarithm is required : the Difference of their Logarithms nearly ; which being either added to, or subtracted from, the nearest Logarithm, according as it is greater or less than the required one, will give nearly the Logarithm required. Take an Example.

Let it be required to find the Logarithm of 367182.

The fractional Part of the Logarithm of 3671 is by the Table .564784, and of 3672 is .564903. ∴

The 367100 is 5.564784 Nearest N°. 367200
Log. of 367200 is 5.564903 The N°. 367182

—————
Their Dif. 100 & of Log. .00119

—————
Diff. 18

∴ As 100 : .00119 :: 18 : .00021 nearly ; ∴
5.564903 — .00021 = 5.564882 nearly = the Logarithm of 367182.

This Method being grounded on Supposition, that the Logarithms of Numbers between 367100 and 367200 increase or decrease equally, according to their Distance from 367100 or 367200, is not strictly true, but nearly so ; and the greater any Numbers are with Respect to their Difference, the nearer will those Differences be proportional : And therefore, though this will not give the exact Logarithm, yet it will be nearly so, and sufficiently near the Truth for most Purposes either of Business or Pleasure.

If the Number does consist of both Integers and Fractions, find the fractional Part of the Logarithm as if all its Figures were Integers, and to that fractional Part prefix the Index belonging to the integral Part ; viz. make the Characteristick one less than the Number of integral Figures, and you will have the Logarithm sought.

Lastly, if the Number whose Logarithm is required, be entirely fractional ; find the fractional Part of the Logarithm, as if it was a whole Number, and

and to it prefix the Characteristick -1 , if the first Figure to the left Hand be *Tenths*; but, if the first Figure be in the second Place, or Place of *Hundreds*, the Characteristick must be -2 , &c.

23. When two or more Logarithms are to be added together, they may be added by the Rules for adding affirmative and negative Quantities in Algebra. That is, 1. If they are all *affirmative*, add them as in common Addition, and the Sum will be the affirmative Logarithm required.

24. *Case 2.* If the Characteristicks of the Logarithms to be added are all *negative*, add the decimal Parts together by common Addition; and if there is any Thing to be carried from the decimal Part to the integral Parts, it is *affirmative*, (by the Nature of binomial Logarithms,) which reserve: Then find the Sum of the negative Characteristicks; which, being negative, the Difference betwixt this Sum, and the Carriage (above reserved,) with the Sign of the greater, will be the Characteristick required.

25. *Case 3.* When the Characteristicks of the Logarithms to be added, are some *affirmative* others *negative*. — Add the fractional Parts as usual, only the Carriage from the decimal Part to the integral being affirmative, must be added to affirmative Characteristicks; whose Sum, including the Carriage, being found, and also the Sum of the negative Characteristicks, find the Difference of these Sums, and to it prefix the Sign of the greater, for the required Characteristick.

26. When one Logarithm is to be subtracted from another, the Work may be done as in Subtraction of Algebra; by having due Regard to the affirmative and negative Quantities. But to be more particular.

Case 1. To subtract a less affirmative, from a greater affirmative Logarithm; find their Difference, as in common Arithmetick.

27. *Case 2.* When it is required to subtract a greater affirmative, from a less affirmative Logarithm; it is the same as to find the Logarithm of a proper

To find the Difference of any two Logarithms.

proper Fraction, and therefore may be found as directed in Article 21.

28. *Case 3.* When a *binomial* Logarithm is to be subtracted from an affirmative Logarithm, if the decimal Part of the affirmative Logarithm is greater than that of the Binomial, subtract the decimal Part of the *Binomial* from the decimal Part of the Affirmative; and prefix the Sum of their Characteristicks, with the affirmative Sign. But if the *fractional* Part of the binomial Logarithm be greater than that of the Affirmative, add, or suppose in your Mind 1 to be added to the fractional Part of the Affirmative; then subtract the fractional Part of the Binomial from that Sum; (but, because we added 1 for Convenience, we must also to continue the Equality) add $+1$ to the binomial Characteristick; but the Characteristick of the binomial Logarithm being negative, the adding $+1$ to it is diminishing the negative Characteristick by Unity; which Characteristick so diminished, being added to the affirmative Characteristick, will give the affirmative Characteristick of the required Logarithm.

29. *Case 4.* When an *affirmative* Logarithm is to be subtracted from a *Binomial*; if the fractional Part of the Binomial be greater than that of the affirmative Logarithm; to the Difference of their fractional Parts, place for a Characteristick the Sum of their Characteristicks with the negative Sign: But if the *decimal* Part of the Binomial is less than that of the affirmative Logarithm, add 1 (or suppose it to be added in your Mind) to the fractional Part of the Binomial; and then subtract the decimal Part of the Affirmative from that Sum, and (to continue the Equality) add 1 to the Characteristick of the affirmative Logarithm; then the Sum of this, and the Characteristick of the Binomial, will, with the negative Sign, be the Characteristick of the required Logarithm.

30. *Case 5.* When a *binomial* Logarithm is to be subtracted from a *Binomial*; if the *fractional* Part of that

that to be subtracted is less than that from which it is to be taken, find the Difference of their fractional Parts, to which prefix the Difference of their Characteristicks, with the affirmative Sign, if the Characteristick of the Logarithm to be subtracted is the *greater*; otherwise it must have a negative Sign. But if the *fractional* Part of the Logarithm to be subtracted, be greater than the *fractional* Part of the Logarithm from which it is to be taken, add 1 to the fractional Part of this last, that Subtraction may be made; and then, (to continue the Equality, add 1 to the Characteristick of the *Minorand*, which Characteristick being negative, is done by Subtraction, *viz.* subtract 1 from the Characteristick of the *Minorand*; then the Difference between the Characteristick of the *Minorand* so diminished, and the Characteristick of the *Subducend*, will give the required Characteristick; which must be *affirmative*, if the Characteristick of the *Minorand*, after being diminished as above, is greater than that of the *Subducend*: Otherwise *negative*.

31. Before we proceed to the Application of Logarithms, it will be proper to shew how to find the *natural* Number answering to any given Logarithm, by Help of a Table of Logarithms. And as this is only the Reverse of Article 22, it will be sufficient to hint, 1st, That the Index, or Characteristick, of the Logarithm, if *affirmative*, will shew how many integral Places are in the required Number: And if *negative*, in what Place of Decimals the first, or significant Figure, stands. Hence, if the Logarithm can be found in the Table, the answering Number is found by a bare Inspection: But if it cannot be exactly found, find the next greater, and next lesser, and say, As the Difference of these two Logarithms: the Difference of the answering Numbers :: the Difference between the given Logarithm and the nearest tabular Logarithm: a fourth Number; which added to, or subtracted from, the natural Number answering to the nearest tabular Logarithm, (according

Multiplication by Logarithms.

(according as that Logarithm was less or greater than the given one,) will give the required Number very near the Truth.

Example. Let it be required to find the natural Number answering to the Logarithm 5.564882.

This in the Tables gives the next lesser and greater $\left\{ \begin{array}{l} 5.564784 \\ 5.564903 \end{array} \right\}$ The answering $\left\{ \begin{array}{l} 367100 \\ 367200 \end{array} \right\}$ Logarithms, Numbers

Their Diff. .000119

100

And $5.564903 - 5.564882 = .000021$. Hence, As .000119 : 100 :: .000021; or, which is the same in Effect, As 119 : 100 :: 21 : 17.7 nearly. Hence, $367200 - 17.7 = 367182.3$, nearly = the required Number.

C H A P. II.

MULTIPLICATION *by* LOGARITHMS.

32. **W**E now come to shew the Application of Logarithms, having explained their Theory at large; and the first Thing is to perform Multiplication by them; which is done by adding the Logarithms of the Factors together, as directed in Art. 23, 24, and 25.

33. *Example 1.* Multiply 14 by 4.

Log. of $\left\{ \begin{array}{l} 14 \text{ is } 1.1461280 \\ 4 \text{ is } 0.6020600 \end{array} \right\}$

Their Sum 1.7481880 answers in

the Table to 56, the required Product.

Multiplication by Logarithms.

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34. *Example 2.* Multiply 203 by 0.25.

Log. of 203 is 2.3074960

Log. of 0.25 is 1.3979400

This Log. 1.7054360 answers in the Table to
59.75.

35. *Example 3.* Multiply 7.677 by $\frac{1}{2}$.

To Log. of 7.677 = 0.8851915

Add Log. of $\frac{1}{2}$ = 0.6989700

From 1.5841615

Subtract Log. of 8 = 0.9030900

Gives Log. of 4.798 nearly, 0.6810715

Or thus: Put $\frac{1}{2}$ into Decimals, viz. $\frac{1}{2}$ = .625.

Then, Log 7.677 = 0.8851915

Log. of $\frac{1}{2}$ or of .625 = 1.7958800

Sum is 0.6810715 as above.

36. *Example 4.* Multiply 0.55 by .023.

Log. .55 is 1.7403627

Log. .023 is 2.3617278

Log. of .01265 2.1020905

37. *Example 5.* Multiply 80 by .003.

Log. 80 = 1.9030894

Log. .003 = 3.4771212

Log. .24 = 1.3802116

Multiplication by Logarithms.

38. *Example 6.* Multiply 10, 80, 72, .5, .4, and .12, together.

Log. 10 = 1.0000000	Log. .5 = $\overline{1.6989700}$
Log. 80 = 1.9030894	Log. .4 = $\overline{1.6020600}$
Log. 72 = 1.8573325	Log. .12 = $\overline{1.0791812}$
Tp 4.7604219	
Add $\overline{2.3802112}$	$\overline{2.3802112}$

This Log. 3.1406331 answers in the Table to
[1382.4 nearly.]

C H A P. III.

DIVISION by LOGARITHMS.

39. **T**HE Subtraction of Logarithms answering to the Division of the natural Numbers, from the Logarithm of the Dividend subtract the Logarithm of the Divisor, by the Rules in Art. 26, 27, 28, 29, 30; and the Remainder will be the Logarithm of the required Quotient.

40. *Example 1.* Divide 56 by 4.

Log. 56 = 1.7481880
Log. 4 = 0.6020600
Log. 14 = 1.1461280

41. *Example 2.* Divide 5 by 8.

Log. 5 = 0.6989700
Log. 8 = 0.9030900
Log. .625 = $\overline{1.7958800}$

Division by Logarithms.

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42. *Example 3.* Divide 50.75 by .25.

$$\text{Logarithm of } 50.75 = 1.7054360$$

$$\text{Log. of } .25 = \overline{1.3979400}$$

$$\text{Log. of } 203 = \underline{2.3074960}$$

43. *Example 4.* Divide 4.718 by .625.

$$\text{Log. of } 4.718 = 0.6737579$$

$$\text{Log. of } .625 = \overline{1.7958800}$$

$$\text{Log. of } 7.549 \text{ ferè} = \underline{0.8778779}$$

44. *Example 5.* Divide .75 by 100.

$$\text{Log. of } .75 = \overline{1.8750613}$$

$$\text{Log. of } 100 = \underline{2.0000000}$$

$$\text{Log. of } .0075 = \overline{3.8750613}$$

45. *Example 6.* Divide .24 by 80.

$$\text{Log. of } 24 = \overline{1.3802112}$$

$$\text{Log. of } 80 = \underline{1.9030900}$$

$$\text{Log. of } .003 = \underline{3.4771212}$$

46. *Example 7.* Divide .75 by .25.

$$\text{Log. of } .75 = \overline{1.8750613}$$

$$\text{Log. of } .25 = \overline{1.3979400}$$

$$\text{Log. of } 3 = \underline{0.4771213}$$

Division by Logarithms.

47. *Example 8.* Divide .01265 by .55.

$$\text{Log. of .01265} = \overline{2}.1020905$$

$$\text{Log. of .55} = \overline{1}.7403627$$

$$\text{Log. of .023} = \overline{2}.3617278$$

C H A P. IV.

Of the APPLICATION of LOGARITHMS to
CIRCULATING DECIMALS.

48. **A**S in some Calculations, in which we propose to be very accurate, we may have Occasion to find the *Logarithms of circulating Decimals*; we shall explain the Method by which they may be found, as follows.

The RULE. First turn the *circulating* Decimal into its equivalent vulgar Fraction, (as shewn in Theorems 4, 5, 6, and 7, in Articles 13, 14, 15, 16, 17, 18, 19, 20, and 21, of the *Essay on Circulating Decimals*;) and then find the Logarithm of that vulgar Fraction, by Art. 21 of this ESSAY.

49. *Example 1.* What is the Logarithm of $\dot{4}$?

By Art. 14. of the *Essay on Circulating Decimals*,

$$\dot{4} = \frac{4}{9}.$$

$$\text{Hence from Log. of 4} = 0.6020600$$

$$\text{Subtract Log. of 9} = 0.9542425$$

$$\text{Gives Log. of } \frac{4}{9} \text{ or of } \dot{4} = \overline{1}.6478175$$

50. *Example 2.* What is the Logarithm of $\dot{3}.5$?

Solution. By Art. 16 of the *Essay on Circulating Decimals*, $\dot{3}.5 = \frac{35}{9}$.

From Log. of 350 = 2.5440680

Subtract Log. of 99 = 1.9956352

Gives Log. of $\frac{35}{9} = 0.5484328$

51. *Example 3.* What is the Logarithm of $\dot{.23}$?

Solution. By Art. 18 of the *Essay on Circulating Decimals*, $\dot{.23} = \frac{23}{99}$.

From Log. of 21 = 1.3222193

Subtract Log. of 90 = 1.9542425

Gives Log. of $\frac{23}{99} = 1.3679768$

52. *Example 4.* What is the Log. of $26\dot{3}53.5$?

Solution. By Art. 20 of the *Essay on Circulating Decimals*, $26\dot{3}53.5 = \frac{2609000}{99}$.

Therefore, from Log. of 2609000 = 6.4164741

Subtract the Log. of 99 = 1.9956352

The Remainder is the Log. of $\frac{2609000}{99} = 4.4208389$

53. The Logarithms of Circulating Decimals being found by the above Method; Multiplication, and Division, are performed by them, after the same Manner as is shewn of finite Numbers in Chap. 2. and 3. of this *ESSAY*.

C H A P. V.

The GOLDEN RULE by LOGARITHMS.

54. **F**ROM the Nature of the *Golden Rule*, and Logarithms, we have this Rule. Add the Logarithms of the second and third Terms together; from their Sum subtract the Logarithm of the first Term; and the Remainder will be the Logarithm of the fourth, or required Term.

55. Suppose in the Rule of Three Direct we have this Stating, As 4.1 : 5 :: 65 : a fourth Number.

To the Log. of 5 = 0.6989700

Add the Log. of 65 = 1.8129134

2.5118834

Sub. Log. of 4.1 = 0.6127838

Log. of 79.26 nearly 1.8990996

55. If from the Logarithm of 1 be subtracted the Logarithm of any Number, the Remainder is called the *Arithmetical Complement* of that Logarithm. Hence, the *Arithmetical Complement* of the Logarithm of any Number, is equal to the Logarithm of the Quotient of 1 divided by that Number.

56. When it is required to divide one Number by another, it is in Effect the same, if we divide 1 by the Divisor, and multiply that Quotient by the given

Dividend. (For $\frac{a}{m} = \frac{1}{m} \times a$.) Which is done by

Logarithms, by adding the Logarithm of the Dividend to the *Arithmetical Complement* of the Logarithm of the Divisor.

57. Hence, the Operation of the Example in Art. 55, may stand thus :

From

From Log. of 1 = 0.0000000

Sub. Log. of 4.1 = 0.6127839

1.3872161

0.6989700

Gives *Arith. Comp.* = 1.3872161

1.8129134

Gives Log. as above 1.8990995

CHAP. VI.

INVOLUTION by LOGARITHMS.

58. **T**O raise any Number to a given Power by Logarithms, multiply the Logarithm of the Root by the Index of the given Power. Thus, for Instance, if we are to square any Number, we must multiply its Logarithm by 2; if to cube a Number, by 3, &c. and the Product (by Art. 4.) will be the Logarithm of the required Power.

Note, If the Root be a Fraction, its Logarithm will be a *Binomial*; in such Case, the Carriage from the decimal Part in multiplying being affirmative, and the Product of the Characteristick negative, the Carriage must be subtracted from that negative Product, for the required negative Characteristick.

59. *Example 1.* What is the Square of 15?

Log. of 15 = 1.1760913

2

Log. of 225 = 2.3521826

60. *Example 2.* Cube .55.

Log. of .55 = 1.7403627

3

Log. of .166375 = 1.2210881

CHAP. VII.

EVOLUTION by LOGARITHMS.

61. **T**O extract the Root of any Number, divide the Logarithm of the Power by its Index, viz. To extract the square Root, divide by 2; for the Cube Root, by 3, &c. See Art. 5.

Note. In extracting the Root of a fractional Number, if we cannot divide the Characteristick exactly by the Index of the Power, it will be proper to increase the Characteristick by the Addition of such a Number as will make it exactly divisible by the Index of the Power, and then divide the Characteristick so increased; but (since increasing the negative Characteristick by any Number is really diminishing the Logarithm by that Number, we must, in Order to continue the Equality, add it to the Logarithm again; therefore, the next Place of the Logarithm being *Tenths*, each Unit in the Characteristick is equal to Ten in the Place of *Tenths*; therefore,) for every Unit which we added to the Characteristick we must now add 10's to the Place of *Tenths*; and divide as usual.

62. *Example 1.* Extract the Square Root of 225.

$$\text{Log. of } 225 \text{ is } = 2.3521825$$

$$\text{Log. of } 15 = 1.1760912$$

63. *Example 2.* Extract the Cube Root of .166375.

$$\text{Log. of } .166375 = 1.2210881$$

$$\text{Log. of } .55 = 1.7403627$$

Here

Here, to the Characteristick $\overline{1}$, to make it divisible by 3, we add $\overline{2}$, and the Sum is $\overline{3}$; $\frac{1}{3}$ of which is $\overline{1}$, for the Characteristick of the Log. of the Root: Then, to continue the Equality, we add 2 to the Place of Tenths, which makes how many Times 3 in 2.2, which gives .7, &c.

C H A P. VIII.

The APPLICATION of LOGARITHMS to FELLOWSHIP.

64. **A**S in Fellowship, the first and second Terms continue the same in all the Statings: From the Logarithm of the second Term subtract the Logarithm of the first Term, and call the Remainder the *reserved Logarithm*: Then, this *reserved Logarithm* being added to the Logarithm of the third Term of each respective Stating, will give the Logarithm of the fourth, or required Term of that Stating. The Reason of this is manifest from the Nature of Logarithms, and Chap. VIII. of the Second of the *Mathematical Essays*.

65. *Example.* Suppose 4 Men, A, B, C, and D, trade in Company; A put in 50*l.* B, 16*l.* C, 25*l.* and D, 18*l.* 10*s.* they gained 20*l.* 15*s.* What was each Man's Part?

Here the Statings would be,

£.	£.	£.	
As 109.5	: 20.75	:: 50	: A's Share.
109.5	: 20.75	:: 16	: B's Share.
109.5	: 20.75	:: 25	: C's Share.
109.5	: 20.75	:: 18.5	: D's Share.

The Operation by Logarithms will be as follows,

From Log. of £. 20.75 = 1.3170181

Sub. Log. of 109.5 = 2.0394141

The Reserved Log. = 1.2776040

E

From

Fellowship by Logarithms.

$$\text{Log. } 50 = 1.6989700$$

$$\text{Reserved Log.} = \overline{1.2776040}$$

Sum = 0.9765740 the Log.
of A's Part answering in the Table to 9.47485 l.

$$\text{Log. of } 16 = 1.2041200$$

$$\text{Reserved Log.} = \overline{1.2776040}$$

$$\text{Log. of B's} = 0.4817249 \text{ answers}$$

[to £. 3.031952.

$$\text{Log. } 25 = 1.3979400$$

$$\text{Reserved Log.} = \overline{1.2776040}$$

$$\text{Log. of C's Part} = 0.6755440 \text{ answering}$$

[to 4.737425.

$$\text{Log. of } 18.5 = 1.2671717$$

$$\text{Reserved Log.} = \overline{1.2776040}$$

$$\text{Log. of D's Part} = 0.5447757 \text{ answering}$$

[to 3.5056945.

C H A P. IX.

SIMPLE INTEREST by LOGARITHMS.

66. **A**DD the Logarithms of the Principal, Time. and Rate, together; the Sum will be the Logarithm of the Interest.

67. *Example.* What is the Interest of 410 l. 10 s. for 1 Year and 40 Days, at 5 l. per Cent. per Annum?

The Time 1 Yr. 40 Ds. = $\frac{40}{365} = \frac{8}{73}$. Hence,
The Logarithm of the Time is 0.0451622
Log. of 410.5 the Principal = 2.6133132

$$\text{Log. of .05 the Rate} = \overline{2.6989700}$$

Sum = 1.3574454 the Log.
of 22.7743 l. = 22 l. 15 s. 5 d. $\frac{1}{4}$ for the Interest.
Q. E. I.

CHAP.

C H A P. X.

COMPOUND INTEREST *by* LOGARITHMS.

68. **T**HE RULE. Multiply the Logarithm of the Amount of 1*l.* for one Year, by the Time; and to the Product add the Logarithm of the Principal; the Sum will be the Logarithm of the Amount.

69. *Example.* What will 210*l.* 7*s.* 6*d.* amount to in 3 Years, at 5 per Cent. per Annum?

Solution. First, 210*l.* 7*s.* 6*d.* = 210.375*l.* and the Amount of 1*l.* for one Year at 5 per Cent. per Annum = 1.05.

Hence multiply the Log. of 1.05 = 0.0211893

By the Time = 3

0.0635679

Add Log. of 210.375 = 2.3229940

Log. of Amount 243.5353*l.* = 2.3865619

Hence the Amount is 243*l.* 10*s.* 8*d.* $\frac{1}{2}$ nearly.

70. *Scholium.* Mr. Townley, and a few others, instead of the negative Characteristicks write their arithmetical Complements: Thus, for the Log. of .05, they would write 8.6989700, and for the Log. of .005 place down 7.6989700, &c. And the Rule for adding such Logarithms is, If your Characteristicks be negative, add them as in common Arithmetick, only noting, that if the Sum of the Characteristicks be less than 10, add 10; if just 10, add Unity; if more than 10, cast 10 away, the Sum, or Remainder, will be the required Logarithm, according to their Method. But if the Characteristicks are of different Names, viz. one affirmative, the other negative, add them together as in common Addition; but if the Sum be 10, or more than 10, cast 10 away, the Remainder will be affirmative; but if less than 10, negative.

Example.

*Of Mr. Townley's Indices.**Example.* Multiply 203 by 25.

Log. of 203 is 2.3074960

Their Log. of 25 is $\overline{9.3979400}$ Log. of 50.75 = $\overline{1.7054360}$

As for Division by Logarithms, they perform it after this Manner, viz. If one or both be negative, and the Characteristick of the Dividend is least, add 10 to it; but if the Characteristick of the Dividend be the greater, subtract as usual, and the Characteristick of the Remainder in the first Case will be affirmative, in the latter, negative.

Example. Divide .01265 by .55.Log. of .01265 with their Index $\overline{8.1020905}$ Log. of .55 with their Index $\overline{9.7403627}$ Log. of .023 with their Index $\overline{8.3617278}$

Though this last Method, of writing the arithmetical Complements for the negative Characteristicks, may be something sooner learnt, by such Persons as are ignorant of the Nature of affirmative and negative Quantities: Yet we would recommend the former Method as the most rational, and worthy the Notice of the more ingenious Reader.

Thus much is sufficient at present concerning *Logarithms*. More expeditious Ways of constructing Tables, &c. will be shewn hereafter.

The END.

AN
E S S A Y
ON THE
E L E M E N T S
O F
PLANE GEOMETRY.

In which the most useful Propositions of the celebrated Geometricians, both ancient and modern, are clearly demonstrated ; and the Constructions and Uses of the necessary Instruments explained.

By BENJAMIN DONN.

Ὁ Θεὸς αὐτὸν γεωμετρεῖ. PLATO.

P R E F A C E.

GEOMETRY and *Arithmetick*, as *Plato* justly called them, are “the two Wings by which the Minds of Men mount up to Heaven ;” that is, by which they contemplate and discover the Motions and Properties of the *œ*lestial Bodies.

It is an Observation of *Plutarch*, that “God adorneth and layeth out all the Parts of the World according to Quantity, Proportion, and Similitude :” Or, in *Solomon’s* Words, “God disposes all Things by Number, Weight, and Measure.” Hence *Plato*, with Reason, said, *ὁ Θεὸς ἀπὸ γεωμετρίας*, God always acts according to Geometry.)

Geometry taken in its greatest Extent, includes almost every Branch of the Mathematicks ; in which extensive View, what we have said of Mathematicks in general, in our Preface to the *Mathematical Essays*, is applicable to this. But even taken in the narrow Limits to which it is confined in this Essay, it is a necessary Introduction to almost all the Mathematicks, and the practical Sciences depending thereon ; for the Architect, Diallist, Gauger, Engineer, &c. could do nothing without Geometry, it being as it were the Spring which sets them all at Work.

P R E F A C E.

* “ Besides the Aids which every Branch
 “ of the Mathematics derives from Geometry,
 “ the Study of this Science is of infinite Ad-
 “ vantage in the Uses of Life. It is always
 “ good to think and reason right ; and it has
 “ been justly said, that there is no better
 “ practical Logic than Geometry”. It will
 “ serve as the surest Means to give our Rea-
 “ son the first Habitude and Bent of Truth ;”
 and will “ teach us to operate upon Truths,
 “ to trace the chain of them, subtle and al-
 “ most imperceptible as it frequently is, and
 “ to follow them to the utmost Extent of
 “ which they are capable.

“ The *geometrical Spirit* is not so much
 “ confined to *Geometry*, that it cannot be
 “ taken off from it, and transferred to other
 “ Branches of Knowledge. Works of moral
 “ Philosophy, Politics, Criticism, and even
 “ Eloquence, *cæteris paribus*, would have
 “ additional Beauties if composed by *Geome-*
 “ *tricians*. It so strengthens and enlarges the
 “ Mind, that, if we believe *Quintilian*, it
 “ helps to make an Orator. The Order, Per-
 “ spicuity, Distinction, and Exactness, which
 “ have prevailed in good Books for some Time
 “ past, may very probably derive themselves
 “ from this *geometrical Spirit*, which spreads
 “ more than ever, and in some Sort commu-
 “ nicates itself from Author to Author, even
 “ to those who know nothing of Geometry.

As

* Supplement to Dr. *Harris's* Lexicon, under the Article *Geo-*
metry, collected from *Lock* and others.

P R E F A C E.

As to the History of this inestimable Science, we shall for the most Part give it in the Words of Mr. *Thomas Lediard*.

This Science is said to have been invented by the *Egyptians*, on Account of the Inundations of the *Nile*. This Reason might have induced them to cultivate Geometry with a greater Degree of Care and Attention, for distinguishing again each Man's Property of Land, after the Overflowings of the *Nile*; but Mr. *Rollin* thinks its Origin is undoubtedly of more ancient Date. If we may credit *Josephus*, (of which Opinion were also *Pliny*, *Diodorus Siculus*, and *Cicero*,) the *Assyrians* and *Chaldeans* were the first after the Flood who applied themselves to the Study and Practice of the Mathematics. From the *Assyrians* and *Chaldeans* it was transferred into *Egypt* by *Abraham*, who, by the Command of God, left his native Country, and coming thither, found the *Egyptians* to be of a remarkable Disposition for learning mathematical Arts; he therefore communicated to them *Arithmetick*, *Geometry*, and *Astronomy*. The *Egyptians*, flourishing very much in these Sciences, gave Occasion to *Aristotle* to say, that the *Egyptian Priests* were the first *Inventors* of the *Mathematical Arts*, who, by Reason of their Employments, had sufficient Time upon their Hands for making Discoveries in those Arts. From hence these Sciences were carried to *Greece* by *Thales of Miletus*, who travelled into *Egypt* for the Improvement of himself in Learning. He is said to have found out the 5th, 15th, and 26th Propositions of the first
Book

P R E F A C E.

Book of *Euclid*, as also the 2d, 3d, 4th, and 5th of the fourth Book. He began to observe the Equinoxes and Solstices, and was the first who foretold an Eclipse of the Sun, and also one of the Moon to King *Cyrus*; he is therefore justly accounted, and ought to be looked upon, as the first Founder of the *Mathematics* in *Greece*. He flourished 584 Years before *Christ*. (*Vide Stanley's Lives of the Philosophers.*)

Thales was succeeded by *Pythagoras*, a Philosopher of *Samos*, who greatly improved and adorned the *Mathematics*. He found out the 32d, 44th, 47th, and 48th Propositions of the first Book of *Euclid*, but is especially celebrated for the Invention of the 32d and 47th, for which (as *Apollodorus* says) he sacrificed a * *Hecatomb* to the Gods. He first laid open the Doctrine of Incommensurables and the five regular Bodies. He was the first (that we read of) who opened a School wherein Youth might learn those sublime Arts and Sciences. *Pythagoras* was succeeded by *Anaxagoras* of *Clazomene* and *Oenopides* of *Chios*; these were mentioned by *Plato*: and *Aristotle* affirms that *Anaxagoras* wrote a Treatise of *Geometry*, and *Proclus* ascribes the 12th and 23d Propositions of the first Book of *Euclid* to *Oenopides*. After these came *Briſo*, *Antipho*, and *Hippocrates*,

* *Hecatomb* (ἑκατόμῃ from ἑκατόν a hundred, and βῆς Oxen) was a Sacrifice of an hundred Oxen. *Strabo* says this Custom came from the *Lacedemonians*; who having 100 Cities in their Country, each City used to Sacrifice a Bullock every Year. Some derive the Word from ἑκατόν and πῆς a Foot, and hence conclude that a *Hecatomb* consisted only of 25 four-footed Beasts. The Month in which this Sacrifice was made is called *Hecatombæon*.

Hippocrates, who are all celebrated by *Aristotle*, for attempting the Quadrature of the Circle ; but of these *Hippocrates* was by far the most famous ; from a Merchant he became a Philosopher and Geometrician ; he first attempted the doubling of the Cube by two mean Proportionals, and (as *Proclus* observes) was the first who wrote Elements, and digested into Order what Discoveries had yet been made both by himself and others. *Democritus* was admirable in Philosophy and Mathematics, but his Writings were all destroyed, (as some Authors think) by *Aristotle's* ambitious Desire of having no other Writings but his own read. *Plato* (who is reported to have been a Scholar to *Theodorus Cyrenæus*) added great Lustre to the mathematical Sciences : He vastly enlarged Geometry with great and noble Additions ; he illustrated and set off his Books of Philosophy in a mathematical Way, and encouraged whatsoever was admirable either in Mathematics or Philosophy. Over the Door of his Academy was this Inscription in Greek, " Let no one ignorant of Geometry enter here."

Out of *Plato's* Academy proceeded a Number of Mathematicians ; hence proceeded *Leodamus* the *Thracian*, who practising the Analysis received from *Plato*, found out many useful Things by the Help of it ; *Theætetus* the *Athenian*, who is rendered famous as well by *Plato's* Encomiums (who inscribed a Dialogue to his Name,) as by his own Inventions ; 'twas he that wrote the *Elements* and the Inscription of the regular Bodies : *Archytas* also wrote Elements, and his doubling the Cube is mentioned

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tioned by *Eutocius* ; he has also the Honour of being the first who applied Mathematics to Use in Life. *Gellius* reports, that by his Contrivance a wooden Pigeon was made to fly ; he being followed therein by *Dædalus* and others, yielded Matter to the Poets for Fables. *Neoclides* is famous on Account of his Scholar *Leon*, who wrote Elements of the Mathematics, improved them, and made them more fit for the Uses in Life. *Eudoxus* was not inferior to *Leon*, for to him we are indebted (as some Authors affirm,) for the whole 5th and the principal Propositions of the 12th Book of *Euclid*. He first framed the astronomical Hypothesis, and following *Archytas*, derived down the Springs of Geometry to Mechanics. *Amyclas* the *Heracleot*, *Menechmus*, his Brother *Dinostratus*, and *Helicon* of *Cyzium*, all much improved and augmented Geometry. *Theudius* and *Hermotinus* rendered the Elements more perfect and full. *Xenocrates*, the Master of *Aristotle*, was a Man famous for the Mathematics : A Person ignorant of Geometry being desirous to be his Auditor, he answered him thus, “ Go thy Way, for thou wantest the Handles of Philosophy.” These were all *Platonists*.

Aristotle's Books are all full of mathematical Arguments : Two of his Scholars are especially celebrated ; *Eudemus*, who composed a mathematical History, and *Theophrastus*, who wrote two Books of Numbers, four of Geometry, and one of Indivisibles. To *Arisseus*, *Isidore*, and *Hypsicles* we are indebted for the Book of Solids. Lastly, *Euclid* of *Alexandria*, gathering

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ix

ing together the Inventions of others, disposed them into Order, improved and demonstrated them more accurately, and left us those Elements which lay a solid Foundation for all the Branches of mathematical Knowledge. He taught in the Reign of *Ptolemy Lagus*, *A. R.* 454. The two last Books are attributed to *Hypsicles*, and not to him. He died *A. A. C.* 284. He was followed, about 100 Years after, by *Eratosthenes* and *Archimedes*: The Name of the former was very famous, but his Writings are lost. We have many Remains of *Archimedes*, but more are supposed to be lost. *Polybius*, *Plutarch*, and others have delivered us his Inventions. From him proceeded that Saying, *Δὲς πῶ εἰ καὶ τὴν γῆν κινήσω*, i. e. *Let me have where to stand, and I will move the Earth which Way I please.* *Conon* was cotemporary to him: He was a Geometrician and Astronomer, and his Death is lamented by *Archimedes* in his Book of the Quadrature of the Parabola. These were followed by *Apollonius* of *Perga*, who was called the Great Geometrician on Account of his Conics, of which there are now extant seven Books. To him are also ascribed the fourteenth and fifteenth Books of *Euclid*, which were contracted by *Hypsicles*. *Hipparchus* wrote twelve, and *Menelaus* six Books of Subtenses in a Circle; there are also extant three Books of the latter concerning spherical Triangles. *Theodosius* wrote three Books of spherical Triangles, which are pretty well known.

In the Year of *Christ* 70 flourished *Claudius Ptolemy*: He was not only famous in *Astro-*
2
nomy,

P R E F A C E.

nomy, but in *Geometry* also; which (besides other Things written by him) his Books of Subtenses do especially testify. He contracted the six Books of *Menelaus* and twelve of *Hipparchus* into five Theorems.

Plutarch's mathematical Problems are still extant; and many are to this Day sensible of the learned Commentaries of *Eutocius* on *Archimedes*.

The Inventions of *Philo*, *Diocles*, *Nicomedes*, *Sporus*, and *Heron*, concerning doubling the Cube are recited by him. The Genius of the latter was excellent both for *Mechanics* and *Geometry*; his doubling the Cube is likewise recommended by *Pappus* before all others.

The Works of *Ctesibius*, to whom we owe the Invention of Pumps, are celebrated by *Proclus*, *Pliny*, and *Athenæus*. The Name of *Geminus* is not in the lowest Place among *Mathematicians*: *Nicomachus* was famous for arithmetical, geometrical, and musical Monuments; and *Serenus* well known for his two Books concerning the Section of a Cylinder.

Proclus, *Pappus*, and *Theon* are also celebrated among the Ancients: The learned Commentaries upon *Euclid* by *Proclus* shew what a Geometrician he was: The Praises of *Theon* also are deservedly great; but I shall conclude with *Pappus*, the last in Time among the Ancients, who in Reputation ought to be reckoned the first: He was of *Alexandria*, the most fruitful City of great Men, and lived in the Year of *Christ* 400: The two first of the seven Books of his mathematical Collections
are

are lost; the other five abound with so many various and most noble Inventions in many Parts of the Mathematics, that they are esteemed among the chief Monuments of the Ancients which have come down to us.

To particularize the Works of all who have written on Geometry would require a Volume; we shall therefore only give a List of as many Authors as we can recollect; viz. *Contractus* in the Year 1000. *Causanus* 1440. *Regiomontanus* 1464. *Wernerus* 1480. *Barbarus* 1490. *Lurerus* 1500. *Fineus* 1540. *Billingsly* 1570. *Candalla* 1578. *Keckermannus* 1603. *Clavius* 1604. *Severus* 1630. *Cavallerius* 1635. *Ramus* 1636. *Ghetaldus* 1640. *Torricellius* 1644. *P. Gregory* 1647. *Leotandus* 1654. *Fournier*, *Wallis* 1655. *S. Angelus* 1658. *Carragius* 1659. *Fabry* 1669. *Pardie* 1671. *Guarinus* 1671. *Malmshurienfi* 1673. *Bartholinus* 1674. *Dechales* 1674. *Ceva* 1678. *Barrow* 1678. *Henrion* 1683. *D. Gregory* 1684. *Scotus* 1688. *Sturmius* 1689. *Leybourn* 1690. *Oughtred* 1693. *Hugonus* 1698. *Gottignies* 1699. *Guidonus*, *Vivianus* 1701. *Tacquet* by *Whiston* 1703. *Harris* 1705. *Ward* 1706. *Messieurs de Port-Royal* 1711. *Keil* 1728. *Stone* 1731. *Wolffius* 1732. *Martin* 1739. *Clareaut* 1741. *Le Clerc* 1742. *Simpson* 1747. *Muller* 1748. *Halfpenny* 1753. *Cowley*, * *Simson* 1756.

a 2

What

* It is with Pleasure we acknowledge our Obligations to this great Restorer of *Euclid*, (Professor *Simson* of *Glasgow*,) for several valuable Remarks: And therefore the Theorems, &c. which are taken from *Euclid*, are frequently given in the Words of his correct Translation.

P R E F A C E.

What we have chiefly endeavoured in this *Essay*, has been to give clear Demonstrations of the most valuable Propositions of *Euclid* and other celebrated Authors, both ancient and modern; that the Reader might acquire a competent Degree of Knowledge in Geometry, for an Introduction to the numerous Branches of Mathematics and Philosophy, at a moderate Expence of Time and Application.

It is common indeed for Professors to recommend *Euclid* to their Pupils; but though it must be allowed that the Demonstrations of *Euclid* are in general both elegant and accurate, yet if it be considered, that at the Time *Euclid* wrote, it was thought sufficient to understand his Elements to be a celebrated Mathematician; whereas they are now only an Introduction to other Arts: It would not, we think, be right to advise young Men to study *Euclid* thoroughly, as it would take up more Time than they can conveniently spare. Of this the learned and Reverend Mr. *Bedwell* was so sensible, that he says, * “ In what Time, “ think you, may a Man learn all *Euclid*, and “ so by him be made skilful in this Art? By “ himself I know not whether ever or never; “ And with the Help of another, although “ very expert, I will not promise him that he “ shall attain to Perfection in many Years. — “ *Ramus*, my Author, in reading him, confesseth himself often to have been at a Stand; “ often to have lost himself; often to have hit “ on

* In his Preface to his Translation of *Ramus's* Geometry, printed in the Year 1636.

“ on a Rock, when he thought he had
“ touched Land.” — He soon after advises
his Reader by no Means to refuse the As-
sistance of a Teacher, if he can get one, be-
cause if he be not of a great Genius, he can-
not otherwise attain unto any great Perfection,
without much Labour and Loss of precious
Time.

The Method of Demonstration we have
chosen to make use of in this ESSAY, is the
synthetick, that it might be understood inde-
pendent of Algebra, and make the Reader ac-
quainted with the ancient Method of Reason-
ing, which hath been always justly admired for
its Accuracy. As *Euclid's* Demonstrations
(except a few which were corrupted by *Theon*
and some others,) have stood the Test of more
than 2000 Years, we have endeavoured to fol-
low him as nearly as might be consistent with
our Plan; and if we are not mistaken, the
Reader will here find all that is * necessary by
Way of Elements, and may acquire a compe-
tent Degree of Knowledge in much less Time,
and with greater Ease, than could be done by
the Elements of that justly celebrated Au-
thor.

It

* However, perhaps some Readers may find several Propo-
sitions in some modern Authors not taken Notice of in this Essay,
which they may think too valuable to be passed by in Silence: If
so, we would only hint, that being desirous of having the Ele-
ments as short as could be conveniently done, for the Ease of the
young Student: We must beg Leave to refer such Readers to the
End of our intended Essay on Trigonometry, for a Number of
other geometrical Propositions, as we intend in that Place to give
a Collection of many, by Way of Exercise in Geometry, Trigo-
nometry, and Algebra.

P R E F A C E.

It is probable this Work will meet with Censures of opposite Kinds ; some who are bigotted to Antiquity may perhaps condemn it, because we have taken the Liberty to alter the Order and Demonstrations of some of the Propositions ; whilst others who look upon the Ancients with too little Veneration, may blame us for not having taken greater Liberties : But as we have endeavoured to keep clear of each Extreme, the Censures of both these Sorts of Persons will give us no Concern.

Bideford, August 15,
1759.

B. DONN.

A

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A N
E S S A Y
O N
P L A N E G E O M E T R Y.

B O O K I.

The I N T R O D U C T I O N.

1. **G**EOMETRY originally according to its Etymology signified the Art of measuring Land ; but now is the Science which considers the Extension and Magnitude of Bodies in general.

2. Geometry is either Theoretical, or Practical. Theoretical Geometry is the Science of Quantity, Extension, or Magnitude, abstractedly considered : And is what we shall principally treat of in this Essay.

3. Practical Geometry is the Application of Theoretical Geometry to the Conveniences of Life and Commerce, &c. which make up the principal Bulk of the Mathematicks. We shall therefore only just touch on this Part in this Essay ; intending to treat of its several Branches in some future Essays.

4. Geometry may also be divided into *Plane* and *Solid*. Plane Geometry treats of the Properties of
B Lines

Definitions.

Lines and Superficies ; and is what we shall confine ourselves to in this Essay.

5. Solid Geometry considers the Properties of Bodies, with Regard to their Length, Breadth and Thickness. Of which hereafter.

6. *Euclid's* first Definition is,

A *Point* is that which hath no Parts, or which hath no Magnitude.

2.

7. A *Line* is Length without Breadth.

3.

8. The Ends, Bounds, or Extremes of a Line are called *Points*.*

4.

9. A *right Line*, or *straight Line*, is that which lies evenly between its Points ; or which is the shortest Distance from its extreme Points. And a Line, which lies unevenly between its extreme Points, is called a curve Line.

5.

10. A *Superficies*, or *Surface*, is that which hath only Length and Breadth.

6.

11. A *Surface* is bounded or terminated by Lines, or the Extremes of a Superficies are Lines.

7.

12. A *plane Superficies* is that which lieth evenly between its Bounds ; or that in which a right Line joining any two Points taken any where in it will be wholly in that Superficies.

8.

Is useless, and therefore omitted.

9.

13. A rectilineal Angle is the Inclination of two right Lines to one another, meeting in a Point ;
which

* As some have objected to the first Definition, it may not be improper to remark, that hence it follows, a Geometrical Point is not a Quantity, but the End, Term, or Bound of a Quantity ; and therefore indivisible ; and consequently *Euclid's* Definition that a Point hath no Parts, or Magnitude, is very just.

which Point is called the angular Point or Vertex. Or it is the Opening of two right Lines, which meet in a Point.

Note. When there are two or more Angles at the same Point, it is common to express the Angle by three Letters, of which the middle one is at the Vertex, and the other two at the Lines forming the Angle. Thus DBC or CBD, *Plate 1. Fig. 1.* signifies the Angle at B, formed by the Inclination of the Lines DB and CB. And ABD, or DBA the Angle at B, formed by the Meeting of the Lines AB, DB. But if there is but one Angle at a Point, it is generally expressed by the single Letter at that Point.

10.

14. When a right Line, standing on another right Line, makes the Angles on either Side of it equal to each other; each of these Angles is called a *right Angle*. And the right Line which stands on the other is called a Perpendicular to that other. Plate 1.
Fig. 2.

11.

15. An *obtuse Angle* is that which is greater than a right Angle. Fig. 3.

12.

16. An *acute Angle* is that which is less than a right Angle. Fig. 4.

13.

Is useless and therefore omitted.

14.

17. A *Figure* is that which is inclosed by one or more Boundaries.

15.

18 A *Circle* is a plane Figure, contained by one Line, called the Circumference; to which all right Lines drawn from a certain Point within the Figure, are equal to each other. Which right Lines are called the *Radii*. Fig. 5.

Definitions.

16.

19. And that Point within the Figure from which the equal Lines are drawn, is called the *Center*.

17.

20. A *Diameter* of a Circle is a right Line drawn through the Center, and terminated on both Sides by the Circumference.

21. *Scholium*. At the End of this Definition most Copies add, "which divides the Circle into two equal Parts." But this is not properly a Part of the Definition, but a Corollary following from it; the Truth of which plainly appears by supposing one of the Parts into which the Circle is divided to be laid on the other; for it is manifest in such Case they will both coincide, otherwise the right Lines drawn from the Center to the Circumference could not be equal; and if they coincide, they must be equal, by Axiom 8.

18.

22. A *Semicircle* is a Figure contained by a Diameter, and that Part of the Circumference of a Circle cut off by the Diameter.

19.

23. A *Segment* of a Circle is a Figure contained by a right Line and the Circumference cut off by it, (This is not used in the Elements.)

20.

24. *Rectilineal Figures* are such as are contained by right Lines.

21.

25. *Trilateral Figures*, or *Triangles*, by three right Lines.

22.

26. *Quadrilateral Figures* are contained by four right Lines.

23.

27. *Multilateral Figures*, or *Polygons*, by more than four right Lines.

24.

28. An *Equilateral Triangle* is that which has Fig. 6. three equal Sides.

25.

29. An *Isoceles Triangle* is that which has only Fig. 7. two Sides equal.

26.

30. A *Scalene Triangle* is that which has three Fig. 8. unequal Sides,

27.

31. A *right angled Triangle* is that which has one Fig. 9. of its Angles a right Angle.

28.

32. An *obtuse angled Triangle* is that which hath Fig. 8. an obtuse Angle.

29.

33. An *acute angled Triangle* is that which has Fig. 7. three acute Angles.

30.

34. A *Square* is a four sided Figure, whose Sides Fig. 10. are all equal, and Angles all right Angles.

31.

35. An *oblong, rectangle, or right angled Parallelogram*, is that four sided Figure which has all its Angles right Angles, but not all its Sides equal.

32.

36. A *Rhombus* is that which hath four equal Sides, but its Angles not right Angles.

33.

37. A *Rhomboides* is that which has its opposite Sides equal to one another, but all its Sides are not equal, nor its Angles right ones.

Note. For a Definition of a *Parallelogram*, see Art. 97.

38. *Scholium*. In most Copies of *Euclid*, there is another Condition added, viz. "that its opposite Angles are equal." But this is not necessary, for we shall demonstrate hereafter, that all quadrilateral Figures whose opposite Sides are equal, have also their opposite Angles equal. (See Art. 98.)

Postulates.

34.

39. All other quadrilateral Figures besides these are called *Trapeziums*.

35.

Fig. 14.

40. If two Lines AB, CD, meet any other Line BD, so as to make the Angles a and b , at the same Side equal, they are called *parallel Lines*.

N. B. We have here given a different Definition from *Euclid*, because it will render some of our Demonstrations more compendious, &c. However its Agreement with *Euclid's*, that two parallel Lines if infinitely produced could never meet, is demonstrated in Article 96.

POSTULATES.

1.

41. Let it be granted that a right Line may be drawn from any one Point to another.

2.

42. That a finite right Line may be produced to any Length in a right Line.

3.

43. And that a Circle may be described from any Center, at any Distance from that Center.

Scholium. And that Distance is called the *Radius*.

4.

44. That from a given Point and right Line may be drawn, either parallel to another right Line; or so as to make any given Angle with that other right Line.

N. B. In demonstrating the following Theorems we shall not refer to these Postulates; because the Learner will readily see where they are supposed.

AXIOMS.

AXIOMS.

1.

45. Things equal to one and the same Thing, or to equal Things, are equal to each other.

2.

46. If to equal Things, equal Things be added, the Whole will be equal.

3.

47. If from equal Things, equal Things be taken away, the Remainders will be equal.

4.

48. If equal Things be added to unequal Things, the Wholes will be unequal.

5.

49. If equal Things be taken from unequal Things, the Remainders will be unequal.

6.

50. Things which are double to one and the same Thing, are equal to one another.

7.

51. Things which are halves of the same or equal Things, are equal to one another.

8.

52. Magnitudes which coincide, that is, which exactly fill the same Space, are equal to one another.

9.

53. The Whole is greater than its Part.

10.

54. Two right Lines cannot inclose a Space

11.

55. All right Angles are equal to one another.

12.

56. This seems to be improperly placed amongst the Axioms, and has been the Cause of much Debate amongst both the ancient and modern Geometricians; for which Reason it is here omitted, and a more useful one inserted, *viz.*

All

Characters.

All the Parts taken together are equal to the Whole. *Note.* This is not in *Euclid*, though that illustrious Author frequently makes Use of it.

Characters.

57. For Brevity the following Signs are made use of.

Signification.

+	<i>Plus, or more, or and, or to be added.</i>
—	<i>Minus, or less, or that the Quantity following is to be subtracted, or taken from another.</i>
×	<i>A Rectangle, thus $AB \times BD$ denotes the Rectangle contained under the Length and Breadth denoted by the Lines AB and BD. But we generally express a Square by putting the Figure 2 over the Letter, or Letters, representing its Side. Thus, AB^2 signifies the Square whose Side is AB, or which is the same the Rectangle $AB \times AB$.</i>
:	<i>Is to, or to, } When four Quantities A, B,</i>
::	<i>So is. } C, D, are proportional, that is, when as A is to B so is C to D; we generally write them thus, $A : B :: C : D$. But when three Quantities A, B, C, are proportional the second is repeated thus, as $A : B :: B : C$.</i>
=	<i>Equal to.</i>
Γ	<i>The Greek Letter Gamma denotes greater than.</i>
Λ	<i>The Greek Letter Lambda in a vertical Position signifies less than; but,</i>
∠	<i>In an horizontal Position signifies Angle; and with the Addition of the Letter s, as ∠'s denotes Angles.</i>
	<i>In a vertical Position signifies Parallel.</i>
Δ	<i>Signifies Triangle, and with the Addition of the Letter s, as Δ's, is for Triangles.</i>
R	<i>Radius.</i>

Rt.	Right.	Thus, Rt. $\angle$. is for Right Angle.
$\perp$	Denotes a Perpendicular.	

When a Number is prefixed to any Quantity, it shews how many Times that Quantity is to be taken or repeated : Thus, 6 A, denotes six Times the Quantity signified by A.

In the following Propositions, for *Right Line*, we shall only write, *Line*.

Though we shall make Use of the algebraic Signs as above explained, yet we shall give Demonstrations purely geometrical ; for, to use the Words of * Mr. *Simpson*, (who is justly celebrated for his great Skill in mathematical Learning,) “ It is not the Use of
 “ Symbols, (which some more scrupulous than discerning have condemned,) but the Ideas annexed
 “ to them, that render the Considerations *geometrical* or *ungeometrical*. In pure Geometry Regard
 “ is always had to the absolute Quantity of some
 “ one of the three Kinds of Extension, abstractedly
 “ considered ; and whatever Symbols are used here,
 “ are to be considered as expressive of the Quantities themselves, and not as any Measures, or numerical Values of them. Thus, by $A \times B$, taken in a geometrical Sense, we have an Idea, not
 “ of the Product of two Numbers, (as in the algebraic Notation,) but of a real, rectangular
 “ Space, contained under two Right Lines, represented by A and B, and two others equal to them.

“ So likewise, $\frac{B \times C}{A}$ is not to be understood here
 “ in the Light of an algebraic Fraction, but as a
 “ Right Line, which is a fourth Proportional to
 “ three Right Lines represented by A, B, and C.—
 “ These Distinctions are absolutely necessary to those
 “ who would have an accurate Idea of the Subject.”

* In his select Exercises.

Pl. I. Fig. 15. 58. THEOREM 1. *If two Triangles, ABC, DEF, have two Sides of the one, AB, AC, equal to two Sides of the other, DE, DF, each to each, viz. AB equal to DE, and AC equal to DF, and the Angles contained between those equal Sides equal, viz. the Angle A equal to the Angle D; they shall have the other Side BC equal to the other Side EF, the Triangle ABC equal to the Triangle DEF, and the other Angles in one Triangle shall be equal to their corresponding ones in the other Triangle; each to each, viz. those to which the equal Sides are opposite; that is, the Angle B equal to the Angle E, and the Angle C equal to the Angle F.*

For if the $\triangle ABC$ be applied to DEF, so that the Point A may coincide with D, and the Line AB with DE, the Point B will coincide with E, because $AB = DE$; and the Side AC will coincide with DF, because $\angle A = \angle D$; and AC being $= DF$, the Point C will fall upon the Point F. Hence the Points, B and C, coinciding respectively with the Points E and F, the Base BC must be $=$ the Base EF, otherwise two Lines would contain a Space, which is ^a absurd. $\therefore$ the Side BC coincides and is $=$ the Side EF; $\therefore$ the whole $\triangle ABC$ coincides and is $= \triangle DEF$; $\therefore \angle B$ coincides and is $= \angle E$; and $\angle C$ with and $= \angle F$. Q. E. D.

Pl. I. F. 16. 59. THEOREM 2. *The Angles B, C, at the Base of an Isosceles Triangle ABC, and opposite to the equal Sides AC, AB, are equal to each other; and a Line AD bisecting the Angle BAC, divides the Base into two equal Parts, and is perpendicular thereto.*

For because the Line AD bisects the $\angle A$, that is, makes the $\angle BAD = \angle CAD$, and Side AB is $=$ AC, and AD common to both, the $\triangle ABD =$ ^a $\triangle ACD$; $\angle B = \angle C$, and $\angle ADB =$ ^b $\angle ADC$; $\therefore AD \perp$ to BC. Q. E. D.

60. COROLLARY. Hence every equilateral Triangle is also equiangular.

For the Sides AB, AC, being equal, from above, $\angle B = \angle C$; and if the Δ be equilateral, the Side CA = CB, and $\therefore$ by this Theorem, the opposite $\angle$ s are equal, viz. $\angle A = \angle B \therefore \angle A = \angle B = \angle C$. Q. E. D.

^b 1 Ax.

61. SCHOLIUM. This Theorem is the fifth Proposition of the first Book of *Euclid*, and is by many called the *Asses Bridge*, because those who are not capable of learning that Book, generally stop at this Place; but as *Euclid's* Demonstration is certainly something too difficult to be so soon entered on, we have taken the Liberty to give a different, and it is supposed, a much easier Demonstration.

62. THEOREM 3. If two Angles B, C, of a Tri-Pl. 1. Fig. angle ABC, be equal, then the Sides AB, AC, subtending the equal Angles, will be equal between themselves.

For, if AB be not = AC, one of them is Γ the other. Let AB be the greater, and let DB be = AC. Then, because in the Δ s DBC, ACB, we have DB = AC, BC common to both, and the contained $\angle$ s equal, viz. $\angle B = \angle C$, the Δ DBC will in every Respect be =^a Δ ACB, the less = the greater, which is absurd. $\therefore$ AB is not unequal to AC, $\therefore$ AB must be = AC. Q. E. D.

^a 58.

63. COROLLARY. Hence every equiangular Δ is also equilateral.

For, by this Theorem, the $\angle B$ being = $\angle C$, the Side AB will be = AC; and in an equiangular Δ , the $\angle A = \angle C$, $\therefore$ the opposite
C 2 Sides,

Sides, BC, AB, are equal. $\therefore AC = BC = AB$, by Axiom 1st. Q. E. D.

Pl. I. F. 18. 64. THEOREM 4. *A Line AB standing on another Line DC, making with it Angles, ABD, ABC, those Angles taken together are equal to two Right Angles.*

For, let EB be a $\perp$ to DC, then the $\angle$ s, EBD, EBC, are ^a Right Angles. But $\angle ABD =$ ^b $\angle EBD + \angle EBA$; and $\angle EBA + \angle ABC = \angle EBC$; that is, the $\angle ABD =$ a Rt. $\angle + \angle EBA$, and $\angle EBA + \angle ABC =$ a Rt. $\angle$.

$\therefore$ by adding those equal Things together, viz. the Antecedents to the Antecedents, and the Consequents to the Consequents, we have $\angle ABD + \angle EBA$ ^c $+ \angle ABC = 2$ Rt. $\angle$ s $+ \angle EBA$; and by taking away the $\angle EBA$, common to both Sides of the ^d Equation, we find $\angle ABD + \angle ABC = 2$ Rt. $\angle$ s. Q. E. D.

65. COROLLARY 1. *Hence if one of the $\angle$ s be a Rt. one, the other is also a Rt. $\angle$; and if one of them, ABC, is acute, the other, ABD, is obtuse, et contrà.*

66. COROLL. 2. *If two or more Lines stand on the same Line at the same Point, the Angles which they make, taken together, will be $= 2$ Rt. $\angle$ s; for all the Parts are equal to the Whole.*

67. COROLL. 3. *Two Rt. Lines crossing each other, make 4 $\angle$ s, which taken together are $= 4$ Rt. $\angle$ s.*

68. COROLL. 4. *All the $\angle$ s made about a Point are $= 4$ Rt. $\angle$ s, by Coroll. 2.*

Pl. I. F. 19. 69. THEOREM 5. *If two Right Lines, AB, CD, intersect each other, the opposite Angles are equal, viz. $\angle AEC = \angle BED$, and $\angle CEB = \angle AED$.*

For

For $\angle AEC + \angle CEB = 2 \text{ Rt. } \angle s$, and $\angle CEB + \angle BED = 2 \text{ Rt. } \angle s$; $\therefore \angle AEC + \angle CEB = \angle CEB + \angle BED$; from which taking away the $\angle CEB$, common to both, there remains $\angle AEC = \angle BED$. After the same Way it may be shewn that $\angle CEB = \angle AED$. Hence the opposite $\angle s$ are equal. Q. E. D.

70. THEOREM 6. *Any two Sides of a Triangle are together greater than than the third Side.*

For the nearest Distance to any two Points being $Pl. I. F. 15.$ a Rt. Line, $BA + AC$ must be ΓBC . Q. E. D. $\cdot 9.$

71. Scholium. The Epicureans, as Proclus observes, derided this Theorem, it being manifest to the weakest Capacity, and $\therefore$ required no Demonstration: However, as the Number of Axioms ought not to increased without a Kind of Necessity, Euclid, who is scrupulously exact, thought proper to give a formal Demonstration of it. — What we have said above is sufficient.

72. THEOREM 7. *If a Line EF, crosses two parallel Lines, AB, CD, the alternate Angles, AGH, GHD, are equal to each other; and the two interior Angles, BGH, GHD, upon the same Side, taken together are equal to two Right Angles.* $Pl. I. F. 20.$

For, the Lines AB, CD, being $\parallel$, the $\angle AGH = \angle CHF$; and $\angle GHD = \angle CHF$, $\therefore \angle AGH = \angle GHD$. (After the same Manner it might be shewn that $\angle CHG = \angle HGB$.) Again, it has been just shewn that $\angle AGH = \angle GHD$, $\therefore \angle BGH + \angle AGH = \angle BGH + \angle GHD$. But $\angle BGH + \angle AGH = 2 \text{ Rt. } \angle s$. $\therefore \angle BGH + \angle GHD = 2 \text{ Rt. } \angle s$. Q. E. D.

Pl. 1. F. 20. 73. THEOREM 8. *If a Line EF, falling upon two other Lines, AB, CD, makes the alternate Angles AGH, GHD, equal to each other, the Lines AB, CD, will be parallel.*

For the $\angle$ AGH being $= \angle$ GHD by the Supposition, and $\angle$ AGH $= \angle$ EGB, the $\angle$ EGB $= \angle$ GHD; $\therefore$ AB $\parallel$ CD. Q. E. D.

Pl. 1. F. 21. 74. THEOREM 9. *If two Lines AB, CD, be each of them parallel to another Line EF, those Lines AB, CD, are parallel to each other.*

For AB being $\parallel$ to EF, the $\angle$ AGH $= \angle$ EHI; and the Line CD being $\parallel$ to EF, the $\angle$ EHI $= \angle$ CIK; $\therefore \angle$ AGH $= \angle$ CIK; $\therefore$ AB is $\parallel$ CD. Q. E. D.

75. Scholium. From the bare Definition of parallel Lines, plainly follows this COROLLARY, *that if a Line is $\perp$ to another Line, it is also $\perp$ to all the Parallels of that other Line.* — See Art. 96.

Pl. 1. F. 22. 76. THEOREM 10. *If any Side BC of a $\triangle$ ABC be produced, the external Angle ACD will be equal to the two internal and opposite Angles A and B.*

For let the Line CE be $\parallel$ to BA; then the $\angle$ ECD $= \angle$ B; and, as the Line AC crosses two $\parallel$ Lines BA, CE, the alternate $\angle$ s are equal, viz. $\angle$ ACE $= \angle$ A. $\therefore \angle$ ECD + $\angle$ ACE $= \angle$ B + $\angle$ A. But $\angle$ ECD + $\angle$ ACE $= \angle$ ACD, $\therefore \angle$ ACD $= \angle$ B + $\angle$ A. Q. E. D.

77. COROLLARY. *Hence the external Angle ACD is greater than either of the internal and opposite Angles A, or B; because it is, by this, equal to both.*

78. THEOREM II. *The three (interior) Angles of every Triangle are equal to two Right Angles.* Pl. I. F. 22.

For, the $\angle A + \angle B^a = \angle ACD$, and to these equals adding $\angle ACB$, we have $\angle A + \angle B + \angle ACB^b = \angle ACD + \angle ACB$: But $\angle ACD + \angle ACB^c = 2 \text{ Rt. } \angle s$; $\therefore \angle A + \angle B + \angle ACB^d = 2 \text{ Rt. } \angle s$. 76.
46.
64.
45.
Q. E. D.

79. COROLLARY I. *The three Angles of any Triangle, taken together, are equal to the three Angles of any other Triangle taken together.*

80. COROLL. 2. *If in one Triangle, two of its Angles, taken together, are equal to two Angles of another Triangle taken together, in such Case the remaining Angle in one is equal to the remaining Angle in the other Triangle.*

81. COROLL. 3. *If two Triangles have one Angle of the one equal to one Angle of the other, then the Sum of the two remaining Angles of one Triangle is equal to the Sum of the two remaining Angles of the other.*

82. COROLL. 4. *If in any Triangle one of its Angles is a Right Angle, the Sum of the other two is a Right Angle.*

83. COROLL. 5. *Hence, if in a right-angled Triangle one of the acute Angles is given, the other is also known, it being the Difference between the given Angle and a Right Angle.*

84. COROLL. 6. *When the Angle contained between the Sides of an Isosceles Triangle is a Right Angle, the other two Angles are each of them half a Right Angle.*

For

- For one of the Angles being a Rt. $\angle$ by the
^a 82. Supposition, the Sum of the other two is ^a = Rt. $\angle$.
^b 59. But the $\angle$ s at the Base of an Isosceles $\triangle$ are ^b equal;
 $\therefore$ each of them must be $= \frac{1}{2}$ a Rt. $\angle$, both being
 equal to a Rt. $\angle$, as hath been just shewn.

- ^a 60. 85. COROLL. 7. *An equilateral Triangle being ^a equiangular, each Angle must be equal to one Third of two Rt. Angles; or, which is equivalent, two Thirds of one Right Angle.*

86. COROLL. 8. *If in any Triangle, one of its Angles is either a right or an obtuse Angle, the other two will be each of them acute.*

87. COROLL. 9. *All the interior Angles of any rectilineal Figure, together with four Right Angles, are equal to twice as many Right Angles as the Figure has Sides.*

- Pl. I. F. 23. For any rectilineal Figure, ABCDE, can be divided into as many Triangles as the Figure has Sides, by drawing Lines from any Point F within the Figure, to each of its angular Points. And all the
^a 78. Angles of these Triangles are ^a = twice as many Rt. $\angle$ s as there are Triangles, that is, as the Figure has Sides. And the same $\angle$ s are $=$ the $\angle$ s of the Figure, together with the $\angle$ s at the Point F, which is the common Vertex of the Triangles; that is, ^b together with four Rt. $\angle$ s. Therefore this Corollary is true.

- ^b 68. 88. COROLL. 10. *All the exterior Angles of any rectilineal Figure are together equal to four Right Angles.*

- Pl. I. F. 24. Because every interior $\angle$ ABC, with its adjacent exterior $\angle$ ABD, is ^a = 2 Rt. $\angle$ s; $\therefore$ all the interior
^a 64. together, with all the exterior $\angle$ s of any Figure,
 are

are $\equiv$ twice as many Rt. $\angle$ s as the Figure has Sides, that is, by the last Corollary, they are $\equiv$ all the interior $\angle$ s of the Figure, together with four Rt. $\angle$ s. $\therefore$ all the exterior $\angle$ s are $\equiv$ 4 Rt. $\angle$ s.

89. THEOREM 12. *The greater Side of every Triangle is opposite to the greater Angle:* That is, in $\triangle ABC$, let the Side AC be greater than AB, then, I say, the Angle B is greater than the Angle C. Pl. I. F. 25.

For AC being Γ AB by the Supposition, let D be a Point in the Side AC, so that AD be $\equiv$ AB; and suppose B, D, joined. Then, $\angle ADB$ being the exterior $\angle$ of $\triangle BDC$, the $\angle ADB^a \Gamma$ the interior^a 77. and opposite $\angle C$: But AD being $\equiv$ AB, the $\angle ADB^b \equiv \angle ABD$; $\therefore$ as it has been just shewn^b 59. that $\angle ADB$ is $\Gamma \angle C$, its equal $\angle ABD$ must also be $\Gamma \angle C$, and much more must the whole $\angle B$ (which is $\Gamma \angle ABD$) be $\Gamma \angle C$. Q. E. D.

90. COROLLARY. *Hence, in every Triangle, the greater Angle is subtended by (that is, is opposite to,) the greater Side.*

For by the last Article, the greater Side is opposite to the greater Angle; consequently, the greater $\angle$ is opposite to the greater Side. For if any Thing, A, is opposite to another Thing, B, it is manifest that the Thing B must be opposite to the Thing A. — Here again we have taken the Liberty to deviate from Euclid's formal Demonstration.

91. THEOREM 13. *If two Triangles, ABC, DBC, have the same common Base, the two Sides BD, DC, of the inscribed Triangle, taken together, will be less than the two Sides BA, AC, of the circumscribing Triangle: But the vertical Angle BDC of the inscribed Triangle will be greater than the vertical Angle A of the circumscribing Triangle.* Pl. I. F. 26.

D

As

As the Sides BD, DC, are contained within the Sides BA, AC, it is manifest the Sum of the circumscribing Sides BA + AC must be Γ those which they circumscribe, viz. Γ BD + DC : For whatever Thing includes another, must itself be Γ that other. This is so plain, that Monsieur *Clairault*, a celebrated French Geometrician, censures *Euclid* for taking the Pains to demonstrate it. But if a Demonstration be required, as the Number of Axioms ought not to be increased without a Kind of Necessity, it may be done thus. Suppose BD produced to meet the Side AC in E. Then in $\triangle ABE$ we have BA + AE Γ BE, and to each of these adding EC, we find that BA + AE + EC is Γ BE + EC, but AE + EC = AC, $\therefore$ BA + AC Γ BE + EC. Again, in $\triangle EDC$ we know that CE + ED Γ DC, and to each of these adding DB, we find that CE + ED + DB Γ DC + DB; but ED + DB = BE $\therefore$ CE + BE Γ DC + DB. But it has been proved that BA + AC Γ CE + BE, much more then is it Γ DC + DB. Q. E. D.

2. That the $\angle BDC$ is $\Gamma \angle A$ may be thus shewn. The exterior $\angle BDC$ of the $\triangle CDE$ is Γ the interior and opposite $\angle CED$; and the exterior $\angle CEB$ of the $\triangle ABE$ is Γ the interior and opposite $\angle A$. Hence we have proved that the $\angle BDC$ is $\Gamma \angle CED$, (or, which is the same, CEB,) which is $\Gamma \angle A$; much more then must $\angle BDC$ be $\Gamma \angle A$. Q. E. D.

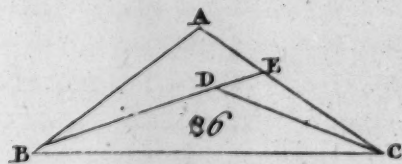
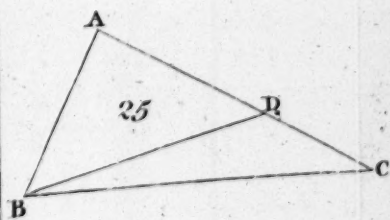
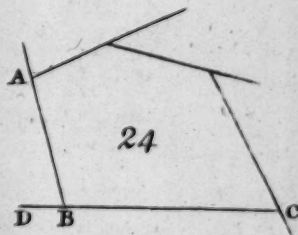
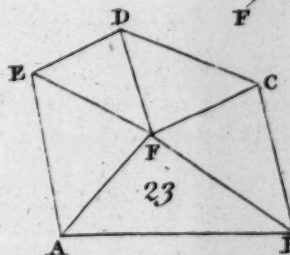
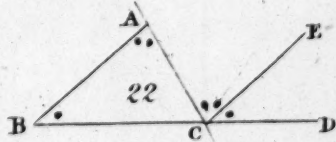
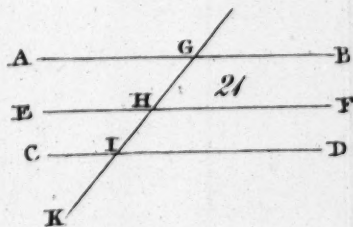
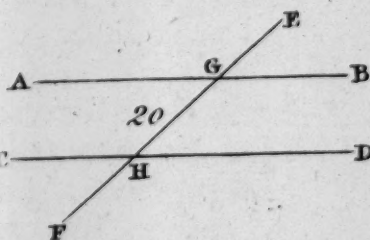
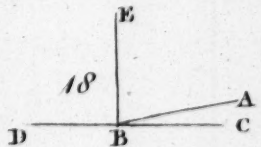
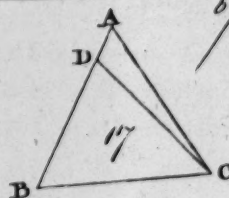
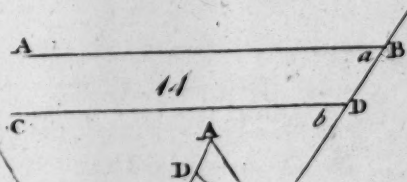
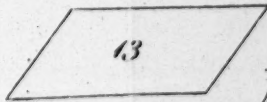
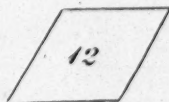
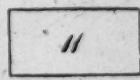
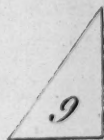
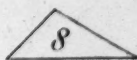
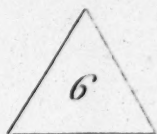
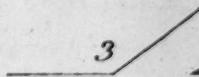
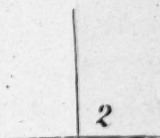
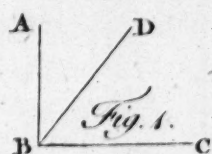
Pl. 2. F. 1. 92. THEOREM 14. If two Triangles ABC, DEF, have two Sides of the one equal to two Sides of the other, each to each, viz. AB = DE, and AC = DF; but the contained Angle of one greater than the contained Angle of the other, viz. the Angle A greater than the Angle EDF; that which has the greatest Angle between the equal Sides will have the greatest Base, viz. BC will be greater than EF.

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Of the two Sides DE, DF, let DE be that which is not Γ the other; and let DG be a Line $=$ DF, making an $\angle$ EDG $=$ $\angle$ A; and let the Points F and G be joined. Then the two Sides AB, AC, of the $\triangle$ ABC are respectively $=$ the two Sides of the $\triangle$ DEG, and $\angle$ BAC $=$ $\angle$ EDG, by the Supposition; $\therefore$ the Base BC $=$ Base EG. And because DG $=$ DF, the $\angle$ DFG $=$ DGF; but the $\angle$ DGF Γ $\angle$ EGF, and $\therefore$ $\angle$ DFG is Γ $\angle$ EGF, and much more shall the $\angle$ EFG be Γ $\angle$ EGF. And the $\angle$ EFG of the $\triangle$ EFG, being Γ $\angle$ EGF, the Side EG is Γ EF: But EG $=$ BC, and $\therefore$ BC must be Γ EF. Q. E. D.

a 58.

b 59.

c 89.

93. THEOREM 15. *If two Triangles ABC, DEF, have two Sides of one, AB, AC, equal to two Sides of the other, DE, DF, each to each, viz. AB $=$ DE, AC $=$ DF; but the Base of one, BC, greater than the Base of the other; the Angle contained by the equal Sides of that which has the greater Base, shall be greater than the Angle contained by the equal Sides of the other; viz. the Angle A greater than the Angle D.* Pl. 2. F. 2.

This is manifest from the preceding Article, being only the Reverse of that; for if that which has the greatest $\angle$ will have the greatest Base, consequently the greatest Base must have the greatest $\angle$. However, Euclid has thought proper to demonstrate it in the following Manner, viz. If the $\angle$ A be not Γ $\angle$ D, it must be either $=$ or Λ $\angle$ D: But it is not equal, for then the Base BC would be $=$ EF, but it is not; $\therefore$ $\angle$ A is not $=$ $\angle$ D. Nor can the $\angle$ A be Λ $\angle$ D; for then the Base BC would be Λ EF, but it is not; $\therefore$ $\angle$ A is not Λ $\angle$ D, and it has been shewn not to be $=$ it; $\therefore$ $\angle$ A must be Γ $\angle$ D. Q. E. D.

a 58.

b 92.

94. THEOREM 16. *If two Triangles have two Angles of one equal to two Angles of the other, each to each,*

D 2

each,

each, and one Side of one equal to one Side of the other, the Triangles are equal in all Respects.

The two $\angle$ s of one being = two $\angle$ s of the other,
 * 80. each to each, the remaining $\angle$ of one ^a = the remaining $\angle$ of the other; and $\therefore$ the three Angles of one = the three $\angle$ s of the other, each to each.
 Pl. 2. Fig. Let ABC, DEF, be the two Δ s, in which Side BC
 3. = Side EF, $\angle B = \angle E$, $\angle C = \angle F$, and $\angle A = \angle D$. Then if the Δ DEF be supposed to be laid on Δ ABC, so that Side EF agrees with its equal BC; then, since $\angle B = \angle E$, the Side ED will coincide with the Side BA, and $\therefore$ the Point D will be somewhere in Line BA; and $\angle F$ being = $\angle C$, Side DF will coincide with CA, and $\therefore$ Point D must be somewhere in Line CA; and consequently, as the Point D is both in the Line BA and CA, it must be in the Point where they meet, viz. in the Point A, and so $DE = BA$, $FD = CA$, and the Δ s in every Respect = each other. Q. E. D.

Note. Euclid having demonstrated this Proposition, by proving the contrary to be absurd, we imagined this would be more acceptable, as it is both direct and easy to be understood.

95. THEOREM 17. If two Triangles have the three Sides of one equal to the three Sides of the other, each to each, the Angles of one will be equal to the Angles of the other, each to each: Or, the Triangles are equal in all Respects.

Pl. 2. F. 3. For AB being = DE, and AC = DF, if the $\angle$ s
 A, and D, be one Γ the other, the Bases BC, EF,
 * 92. will be one ^a Γ the other; but they are equal by the Supposition: $\therefore \angle A$ must be equal to $\angle D$, and $\therefore$
 * 58. the Δ s ABC, DEF, are ^b equal in all Respects; and consequently the $\angle A = \angle D$, $\angle B = \angle E$, and $\angle C = \angle F$. Q. E. D. — This Demonstration is also different from Euclid's, as are some others not taken Notice of.

96. THEOREM 18. *Two parallel Lines AB, CD, though indefinitely continued, would never meet; or, in other Words, they continue to be at the same Distance from each other.*

Let GF be a Perpendicular to the Line AB, and HE $\perp$ to CD, and suppose the Points F, E, joined; then the Δ s EGF, FHE, have FE common, $\angle$ GEF = $\angle$ FHE, and $\angle$ s G, and H, each Rt. $\angle$ s; and $\therefore$ the Δ s have 2 $\angle$ s and one Side of one = 2 $\angle$ s and one Side of the other, each to each; and $\therefore$ the said Δ s are equal in all Respects, and consequently the Side GF = Side EH. Q. E. D.

Pl. 2. F. 4.
72.
14.
94.

97. THEOREM 19. *The opposite Sides and Angles of * Parallelograms are equal to each other, and the Diameter (or Diagonal) bisects them, that is, divides them into two equal Parts.*

Let the Parallelogram ABCD be divided into two Parts by the Diagonal BD; then, because AB is $\parallel$ DC, the $\angle$ ABD = $\angle$ BDC; and because AD is $\parallel$ BC, the alternate $\angle$ s are equal, viz. $\angle$ ADB = $\angle$ DBC; $\therefore$ the two Δ s ABD, CBD, have two $\angle$ s ABD, ADB, in one, = two $\angle$ s BDC, DBC, of the other, each to each, and the Side BD common to both Δ s; $\therefore$ Δ s ABD, CDB are equal in all Respects, (and consequently the Parallelogram is divided into two equal Parts;) $\therefore$ AB = DC, and BC = AD; $\angle$ A = $\angle$ C, and by adding equals, $\angle$ ABD + $\angle$ DBC = $\angle$ ADB + $\angle$ BDC, that is, $\angle$ ABC = $\angle$ ADC; but it has been just shewn that $\angle$ A = $\angle$ C, $\therefore$ the opposite Sides, &c. as in the Theorem. Q. E. D.

Pl. 2. F. 5.
72.
94.

* A Parallelogram is a four-sided Figure, whose opposite Sides are parallel. And the Diameter, or Diagonal, is the Line joining two of its opposite Angles. And the Altitude is the perpendicular Distance betwixt the two opposite Sides.

98. THEOREM 20. *All quadrilateral Figures, whose opposite Sides are equal, have their opposite Angles equal.*

Pl. 2 F. 5. Let the four-sided Figure ABCD be divided into two Δ s by the Diagonal BD; then AB being = DC, BC = AD, and BD common to both Δ s, the Δ s ABD, CDB, have the three Sides of one = the three Sides of the other, each to each; and $\therefore$ the Δ s are in all Respects = each other:

* 95. $\therefore \angle A = \angle C$; $\angle ABD = \angle CDB$, $\angle ADB = \angle CBD$, and consequently, by adding Equals, $\angle ABD + \angle CBD = \angle ADB + \angle CDB$, that is, $\angle ABC = \angle ADC$, but $\angle A$ has been just shewn to be = $\angle C$; $\therefore$ the opposite $\angle$ s are equal. Q. E. D.

99. THEOREM 21. *Parallelograms, or any other Figures between the same Parallels, have equal Altitudes.*

This plainly follows from Article 96; for by that Article the Altitude or $\perp$ Distance between two $\parallel$ Lines, wherever taken, are equal.

100. THEOREM 22. *Rectangles contained under equal Sides are equal.*

Pl. 2 F. 6. For suppose the Diagonals BD, FH, to be drawn: Then, because AD = EH, and AB = EF, by the

* 35. Supposition, and the Angles A and E both = right $\angle$ s; in the Δ s ABD, EFH, are two Sides, AD, AB, and $\angle A$ of one = two Sides EH, EF, and $\angle E$ of the other, each to each, and $\therefore$ the Δ s ABD, EFH, are = equal in all Respects. In the same

* 58. Manner it may be shewn that the Δ s BDC, FGH, are equal; and consequently the whole Rectangle ABCD = the whole Rectangle EFGH. Q. E. D.

101. COROLLARY. Hence, if two Squares have the Side of one equal to the Side of the other, the Squares are equal to each other.

102. THEOREM 23. Parallelograms upon the same Base and between the same Parallels are equal.

If the Sides AD, DF, of the Parallelograms ABCD, DBCF, be terminated in one and the same Point D, it is plain that each of the Parallelograms is ^a double of the $\triangle BDC$, and consequently they are ^b equal to each other. Q. E. D. Pl. 2. F. 7.

But, if the Sides AD, EF, opposite to the Base BC of the Parallelograms ABCD, EBCF, be not terminated in the same Point, then, because ABCD is a Parallelogram, AD is ^c = BC; by the same Reason EF = BC; $\therefore$ AD ^d = EF, and $\therefore$ by adding for Fig. 8. and subtracting for Fig. 9. DE, which is common, we have (AD + DE ^e = EF + DE for Fig. 8. and AD - ED ^f = EF - ED for Fig. 9. that is,) AE = DF; AB is also ^g = DC, and $\angle A = \angle FDC$, from the Nature of ^h parallel Lines; $\therefore$ $\triangle$ s ABE, DCF, have two Sides of one AB, AE, and contained $\angle A$ = two Sides DC, DF, and contained $\angle FDC$ of the other, and $\therefore$ the said $\triangle$ s are ⁱ equal in all Respects; $\therefore$ if each $\triangle$ be taken from the whole Figure ABCF, there will remain ABCD ^k = EBCF. Q. E. D. a 97.
b 50.
c 97.
d 45.
e 46.
f 47.
g 97.
h 40.
i 58.
k 47.

103. COROLLARY 1. Hence Parallelograms having the same Base and equal Altitudes, are equal. See Art. 99.

104. COROLLARY 2. Every Parallelogram is equal to a Rectangle having the same Base and equal Altitudes.

105. COROLL. 3. Hence Parallelograms having equal Bases and equal Altitudes, are equal.

106. THEOREM 24. If a Triangle and Parallelogram stand upon the same Base, and are between the same Parallels, the Triangle is half of the Parallelogram.

Pl. 2. F. 5. For let BDC be the Δ , and let AB be $\parallel$ and $=$
 * 97. DC, and AD $\parallel$ and $=$ BC, then is ABCD a ^a Parallelogram, on the same Base and betwixt the same Parallels. Now, one Side BD of the Δ BDC is a Diagonal to the Parallelogram ABCD, and $\therefore$ the Δ BDC $= \frac{1}{2}$ the Parallelogram ABCD: But the
 b 102. Parallelogram ABCD is ^b $=$ any other Parallelogram upon the same Base and between the same Parallels, $\therefore$ the Δ BDC $= \frac{1}{2}$ of any Parallelogram on the
 * 45. same Base and between the same Parallels. Q. E. D.

107. COROLLARY 1. Hence, if a Parallelogram and Triangle have equal Bases, and are between the same Parallels, or, which is the same ^a Thing, have
 * 99. equal Altitudes, the Triangle is Half of the Parallelogram.

108. COROLL. 2. Hence, if a Triangle and Rectangle have equal Bases and Altitudes, the Triangle is Half the Rectangle.

109. COROLL. 3. Hence, if two Triangles have equal Bases and Altitudes, they are equal to each other.

For the Triangles being the Halves of Parallelograms of equal Bases and Altitudes, which Parallelograms are equal, it follows, that if the Wholes (*viz.* the Parallelograms) are equal, their Halves (*viz.* the Δ s) must be equal also.

110. SCHOLIUM 1. On these two last Theorems and their Corollaries are grounded the Methods of measuring, or finding the Areas, or superficial Contents, of Parallelograms and Triangles. But in Order to this it must be first shewn how to find the Area of a Rectangle; which may be thus illustrated. Let AB^{Pl. 2. F. 10.} be a Line which we suppose to be moved towards DC, so as to continue always || to its first Position, viz. let the Point A be moved in the $\perp$ Line AD, and the Point B in the $\perp$ Line BC; then it is manifest by such Motion it will describe a Rectangle. And if we imagine the Line AB to be divided into any Number of equal Parts, when the Line is gone in Length one such Part towards D, viz. got into the Position *ab*, it will have described a Rectangle *AabB*, containing as many little Squares as AB was divided into Parts; and when it gets into the Position *dc*, two such Parts distant from its first Position, it will have described a Rectangle containing twice as many Squares as AB contains Parts: And so when it arrives at DC, it will have described the whole Rectangle ADCB = as many little Squares (each Side = one Part of AB) as there are Units in the Product of the Number of Parts in AB multiplied by the Number of Parts in BC: And this is called *Measuring*, or finding the *Area* of such Figure. Hence appears the Reason why the *Product* of any two Numbers is by *Geometricians* called the *Rectangle*.

SCHOLIUM 2. Hence may be seen the Reason why $3 \times 7 = 7 \times 3$, or generally $A \times B = B \times A$; which is demonstrated algebraically in our *Mathematical Essays*.

III. Hence, since any Parallelogram is equal to a Rectangle of equal Base and Altitude, to measure any Parallelogram we have this Rule. Multiply the Number of Parts into which the Base is supposed to be

E

divided,

divided, by the Number of like Parts contained in its Altitude, and the Product will be the Area required: Thus, for Example. If the Base of a Parallelogram be 6 Feet, and Altitude 2 Feet, then its Area is $= 6 \times 2 = 12$ square Feet.

112. As to *Triangles*, every Triangle being half of a Parallelogram of equal Base and Altitude, Nothing further need be said.

113. THEOREM 25. *The Complements of Parallelograms which are about the Diagonal of any Parallelogram, are equal to each other.*

Pl. 2 F. 11. Let ABCD be a Parallelogram whose Diagonal is AC; and EH, FG, the Parallelograms about AC, that is, thro which AC passes; and BK, KD, the other Parallelograms, which make up the whole Figure, and are therefore called *Complements*. Then will the Complement BK be $=$ the Complement KD. For, because ABCD is a Parallelogram, and AC its Diagonal, the $\triangle ABC^a = \triangle ADC$; and because EKHA is a Parallelogram whose Diagonal is AK, the $\triangle AEK = \triangle AHK$; and for the same Reason $\triangle KGC = \triangle KFC$; $\therefore$ if from the $\triangle ABC$ we take away the $\triangle$ s AEK and KGC, and from the $\triangle ADC$ take away the $\triangle$ s AHK and KFC, the remaining Quantities must be equal, viz. the Parallelogram BK $=$ the Parallelogram KD. Q. E. D.

Pl. 2. F. 12. 114. THEOREM 26. *The Diagonals AC, DB, of any Parallelogram bisect each other.*

For let the Point of their Intersection be denoted
^a 97. by E. Then, because AD is ^a $=$, and $\parallel$ BC, and
^b 72. the alternate $\angle$ s ^b equal, viz. $\angle ACB = \angle DAC$,
 and $\angle CBD = \angle ADB$, the $\triangle$ s AED, BEC, have two
 $\angle$ s DAE, ADE, and one Side AD of one $=$ two
 $\angle$ s ECB, EBC, and one Side BC of the other,
^c 94. each to each; $\therefore$ the $\triangle$ s AED, BEC, are ^c equal
 in

in all Respects, and consequently the Side $AE =$ Side EC , and Side $DE =$ Side BE , that is, the Diagonals divide each other into two equal Parts. Q. E. D.

115. THEOREM 27. *In any right angled Triangle the Square upon the Side, subtending the right Angle, is equal to both the Squares upon the Sides containing the right Angle.*

Let ABC be the Δ . If we suppose the Line CA produced to G , since $\angle CAB$ is a Rt. $\angle$, the $\angle BAG$ will be so too, and $\therefore$ one Side of the Square upon BA will fall on the Line AG , and so with the Line CA will make one Line CG . For the same Reason one Side of the Square on CA will with BA make one Line BH . Now, suppose the Squares described, and the Points $F, C; B, K; A, D; A, E$, joined by Lines; also $AL \parallel BD$. Then because the $\angle DBC = \angle FBA$, each being a right one, $\therefore$ adding the common $\angle ABC$, we have the whole $\angle DBA =$ the whole $\angle FBC$; and the Side $FB = BA$, and $BD = DC$: Hence the Δ s FBC, DBA , have two Sides of one, FB, BC , and contained $\angle FBC =$ two Sides BA, BD , and contained $\angle DBA$ of the other, each to each, and $\therefore \Delta$ s FBC, DBA are $=$ each other in all Respects: But FB, GC are $\parallel$ by the Nature of Squares (for the $\angle FBA = \angle GAB$, being each a Rt. $\angle$, and $\therefore FB \parallel GC$;) $\therefore$ the Square BG and ΔFBC are on the same Base, and between the same Parallels, and $\therefore$ the Square $BG =$ twice the ΔFBC ; and BD, AL , being Parallels, the Parallelogram $BL =$ twice the ΔDBA ; but the Δ s FBC, DBA , have been just shewn to be $=$ each other, consequently the Square $BG =$ Parallelogram BL . After the same Manner it may be shewn that the Square $CH =$ the Parallelogram CL ; $\therefore$ the Square $BG +$ the Square $CH =$ (the Parallelogram $BL +$ the Parallelogram CL , that is $=$) the Square BE . Q. E. D.

116. SCHOLIUM. This is one of the most extensive and valuable Theorems in Geometry. — For one Example of its Use see 1st. of the *Mathematical Essays*, Article 462. — We shall give another, and easier Demonstration after we have treated of *similar Triangles*; when it will also be shewn to be only a particular Case of a more general Theorem.

B O O K II.

D E F I N I T I O N S.

117. Def. 1. **E**Very right-angled Parallelogram is said to be contained by any two of the Lines which contain one of the right Angles.

118. Def. 2. In every Parallelogram either of the Parallelograms about the Diameter, together with the two Complements, is called the *Gnomon*. Thus Pl.2.F.11. the Parallelogram EH, together with the Complements BK, KD, is the Gnomon, which is more compendiously expressed by the Letters BHF, or GED, which are at the opposite Angles of the Parallelograms which make the Gnomon.

Pl.2.F.14. 119. THEOREM I. If a Line AB be divided into any two Parts in C, the Square of the whole Line AB, viz. ABED, is equal to the Squares of the two Parts AC, CB, together with twice the Rectangle contained by the Parts AC, CB.*

Let

* It may be demonstrated algebraically thus. Let $r = AC$, $x = CB$, then $r + x = AB = AD$, and $\therefore$ the Area of the Square ABED* $= r + x \times r + x =$ (by 1st. of the *Mathemat. Essays*, Art. 455.) $r^2 + 2rx + x^2$, which is the same as the Theorem. Q. E. D. — Hence, if $x = r$, by writing r for $x = rr + 2rr + rr = 4rr$, the Corollary.

Let CF be $\parallel$ AD or BE, and let AH be $=$ CB; and let HK be a $\parallel$ to AB or DE. Then, because AB^a $=$ AD, by taking away equals CB, AH, there remains AC^b $=$ DH. Because HG, DF are Parallels, also HD $\parallel$ GF, the opposite $\angle$ s of the Figure HGFD are^c equal to each other; but the $\angle$ D is a Rt. $\angle$ $\therefore$ the other 3 are Rt. $\angle$ s, and so HGFD is a^a Square each Side $=$ AC. By the same Reason CBKG is a Square each Side $=$ CB; and by the like Method of Reasoning ACGH and KEFG are Rectangles whose Sides are $=$ AC and CB. But the Squares DG, GB, and two Rectangles AG, GE, make up the whole Square ABED. Q. E. D.

^a 34.
^b 47.

^c 98.

^a 34.

120. COROLLARY 1. Hence the Square of any Line is equal to four Times the Square of half that Line.

121. THEOREM 2. A Rectangle made by the Sum and Difference of two right Lines is equal to the Difference of the Squares of the said Lines.*

Let AB, AC, represent the two Lines, and AK, AG, their Squares. Let HG be produced to HF, till GF $=$ AB, or, which is the same, till HF $=$ AC + AB. Let FE be $\parallel$ to AH, and let IK be produced to meet FE in E. Then, because HF $=$ AC + AB, the Sum of the Lines, and IK $=$ AH — AI^a $=$ AC — AB $=$ the Difference of the Lines, 'tis plain the Rectangle HE is contained under the Sum HF, and Difference IK, of the Lines AC, AB,

Pl. 2. Fig. 15.

^a 47.

* This Theorem may be easily demonstrated algebraically. Let the greater Line be denoted by a , the lesser by m , then $a + m$ $=$ their Sum, and $a - m$ $=$ their Difference; then their Rectangle is per Art. 111 $=$ $a + m \times a - m$ $=$ by the Operation in the Margin, $a^2 - m^2$. the same as affirmed in the Theorem. Q. E. D.

$$\begin{array}{r} a + m \\ a - m \\ \hline a^2 + am \\ - am + m^2 \\ \hline a^2 - m^2 \end{array}$$

AB, which we are now to shew to be = the Gnomon HDB, the Difference of the Squares AK, AG; which may be easily done, for GF being = AB by the Supposition, = BK, and DG (= CG - CD = AC - AB) = BC, the Rectangles BD, DF, are contained under equal Sides, and are $\therefore$ equal to each other; $\therefore$ to each adding the Rectangle HD common, we have $BD + DH = DF + HD$, that is, the Gnomon HD = the Rectangle HE. Q. E. D.

122. These two Theorems are sufficient for our present Purpose; we shall therefore pass on to the third Book. — We intend hereafter to give algebraic Demonstrations to all the Theorems of this Book.

B O O K III.

123. *Def. 1.* **E**QUAL Circles are those whose Diameters are equal, or from whose Centers the Lines to the Circumferences are equal: Or, in other Words, they are those that have equal Radii.

124. *Scholium.* This is not, strictly speaking, a Definition, but a Theorem, whose Truth is manifest; for if the Circles are laid on one another so that their Centers may coincide, the Circumferences will also coincide, as the Radii are equal, and $\therefore$ the Circles will exactly cover, and be = each other.

125. *Definition 2.* A Line is said to touch a Circle when it meets the Circle in one Point, and being produced does not touch it in any other Point: And such Line is called a Tangent to the Circle.

126. *Definition 3.* Circles are said to touch one another, which meet, but do not cut each other.

127. *Definition 4.* Lines are said to be equally distant from the Center of a Circle, when the Perpendiculars drawn to them from the Center are equal.

128. *Def. 5.* But the Line on which the greater Perpendicular falls, is said to be further from the Center.

129. *Def. 6.* A Segment of a Circle is the Figure contained by a Line and the Circumference it cuts off, (see Pl. 2. F. 16.) and the Line is called the Chord.

130. *Def. 7.* The Angle in a Segment is that which is contained by two Lines drawn from any Point in the Circumference (or Arch) of the Segment to the Extremities of the Chord Line, or Base of the Segment. See Pl. 2. F. 17.

131. *Def. 8.* And an Angle is said to stand upon the Circumference which the Lines that contain the Angle intercept between them.

132. *Def. 9.* The Sector of a Circle is the Figure contained by two Lines drawn from the Center, and the Circumference between them. See Pl. 2. Fig. 18.

133. *Def. 10.* Similar Segments are those in which the Angles are equal, or which contain equal Angles.

134. **THEOREM 1.** *If a Line CD, drawn through the Center E of a Circle, bisect any Chord AB, it will cut it at right Angles. And if the Diameter CD is at right Angles to the Chord AB, it will bisect it.* Pl. 2. F. 19.

For

- For first, Let EA, EB, be joined; then, because
- a 18. $AF = FB$ by the Supposition, and $AE = EB$ being Radii, and FE common, the Δ s AEF, BEF, have three Sides of one = the three Sides of the other,
 - b 95. each to each, and $\therefore$ they are ^b equal in all Respects; consequently the $\angle AFE = \angle BFE$, and $\therefore$ each =
 - c 14. a Rt. $\angle$. Q. E. D.

- Secondly. But if CD is at Rt. $\angle$ s to AB, then,
- d 59. since $AE = EB$, the $\angle EAF = \angle EBF$; but $\angle AFE = \angle BFE$, each being a Rt. $\angle$ per Art. 14. Hence, in the Δ s AFE, BFE, are two $\angle$ s and one Side of one = two $\angle$ s and one Side of the
 - e 94. other, each to each, and $\therefore$ the Δ s are ^e equal in all Respects: Consequently, their Bases FA, FB, are equal to each other. Q. E. D.

Pl. 2. F. 20. * 134. THEOREM 2. *If any Point F, which is not the Center, be taken in the Diameter AD of the Circle ABCD, whose Center is E; then, of all the Lines FB, FC, FG, &c. that can be drawn from F to the Circumference, that FA, in which the Center is, is the longest, and the other Part FD of that Diameter AD is the least; and of the others, that which is nearer to the Line FA is longer than one more distant; thus, FB greater than FC, and FC greater than FG. And from the same Point F there can be drawn to the Circumference only two Lines equal to each other, viz. one on one Side and the other on the other Side of the Diameter AD, making equal $\angle$ s with it.*

- For 1st. Let BE, CE, GE, be joined; then because two Sides of any Δ are ^a Γ the third, BE +
- a 70. EF Γ BF; but BE = ^b AE, $\therefore$ AE + EF (= BE + EF, and $\therefore$) Γ BF. Again, BE being ^b = CE, EF, common, and $\angle BEF \Gamma \angle CEF$, the Base BF of the Δ BEF is ^e Γ the Base CF of the Δ CEF;
 - b 18. and for the same Reason the Base CF Γ Base GF.
 - c 92. Again, because GF + FE is ^a Γ GE, and GE = ED,

ED, GF + FE must be Γ ED, and taking away the Part FE common, GF must be Γ FD. Hence we have proved that AF Γ BF, BF Γ CF, CF Γ GF, and GF Γ FD: Which were to be demonstrated. Secondly. Let DFH be an $\angle =$ GED, then we affirm the Line FH is $=$ FG, and that no other can be equal to it; for that FH $=$ FG may be thus shewn. Because GE $=$ ^b EH, EF common, and $\angle$ GED $=$ DEF by the Supposition, the Base GF ^a $=$ Base FH. Q. E. D. And that no other Line can be $=$ FG, we thus prove. For if possible, let FK be a Line $=$ FG; and FH has been just shewn to be $=$ FG, and $\therefore$ FK would be ^c $=$ FH, that is, the two Lines FK, FH, $=$ each other, though one is more remote from the Line which passes through the Center than the other; which is contrary to what has been proved above, and $\therefore$ FK is not $=$ FH, and $\therefore$ cannot be $=$ to FG. Q. E. D.

135. THEOREM 3. Let ABC be a Circle, and D any Point without it, from which Lines are drawn to the Circumference. Then we affirm, that of those which fall upon the concave Part of the Circumference AEFC, the greatest shall be DA, which passes thro' the Center M; and the nearer to it greater than the more remote, viz. DE Γ DF, and DF Γ DC. But of those which fall on the convex Part of the Circumference, the least is DG, which coincides with the Diameter AG produced; and the nearer to it is always lesser than the more remote, viz. DK Δ DL, and DL Δ DH. And only two equal Lines can be drawn from the Point D to the Circumference, one upon each Side of the least.

Pl. III.
F. i.

For, 1st. Let the Points ME, MF, MC, MK, ML, MH, be joined; then AM ^a $=$ EM, $\therefore$ adding MD common, EM + MD (^b $=$ AM + MD) $=$ AD; but EM + MD ^c Γ DE, $\therefore$ also DA Γ DE. Again, because $\angle$ EMD Γ $\angle$ FMD,

^a 18.

^b 46.

^c 70.

F

EM

* 18.
* 58. $EM = FM$, and MD common to both Δ s, EMD , FMD , the Base DE $\cap$ Base DF . By the same Method of Reasoning $DF \cap DC$. Hence we have shewn that $DA \cap DE$, $DE \cap DF$, and $DF \cap DC$. Q. E. D.

* 70.
* 18. Second. Because $MK + KD = MD$, taking away equals MK , MG , there will remain $KD \cap DG$. And because the ΔMKD is within the ΔMLD , the two Sides MK , KD , taken together must be Δ the two Sides ML , LD , which include them, (for it is manifest a Thing containing must be $\cap$ the Thing contained;) and taking away equals MK , ML , the Remainder $KD \Delta$ the Remainder LD . In the same Manner DL is ΔDH ; $\therefore DG$ is the least, $DK \Delta DL$, and $DL \Delta DH$. Q. E. D.

Thirdly. Let the $\angle DMB = \angle DMK$; then we affirm that the Line $DB = DK$, and that no other Line DN can be $= DK$: For, that $DB = DK$, may be thus demonstrated. Because $MK = MB$, MD common, and $\angle DMB = \angle DMK$ by the Supposition, the Base $DB =$ Base DK . Q. E. D.

And that no other Line DN can be $= DK$, may be thus shewn: DK is $= DB$, as has been just proved; $\therefore$ if DN be also $= DK$, then will DN be $= DB$, that is, the nearer to the least $=$ the more remote, which is contrary to what has been above demonstrated, and consequently impossible: $\therefore DN$ is not $= DB$. Q. E. D.

Pl. 3. F. 2. 136. THEOREM 4. *If a Point D be taken within a Circle, from which there fall more than two equal Lines to the Circumference, viz. DA, DB, DC, that Point D is the Center of the Circle.*

For if it is possible that D is not the Center, let any other Point E be the Center, in the Diameter $FDEG$. Then, if D be not the Center, DG is * the greatest Line from it to the Circumference, and $DC \cap DB$,

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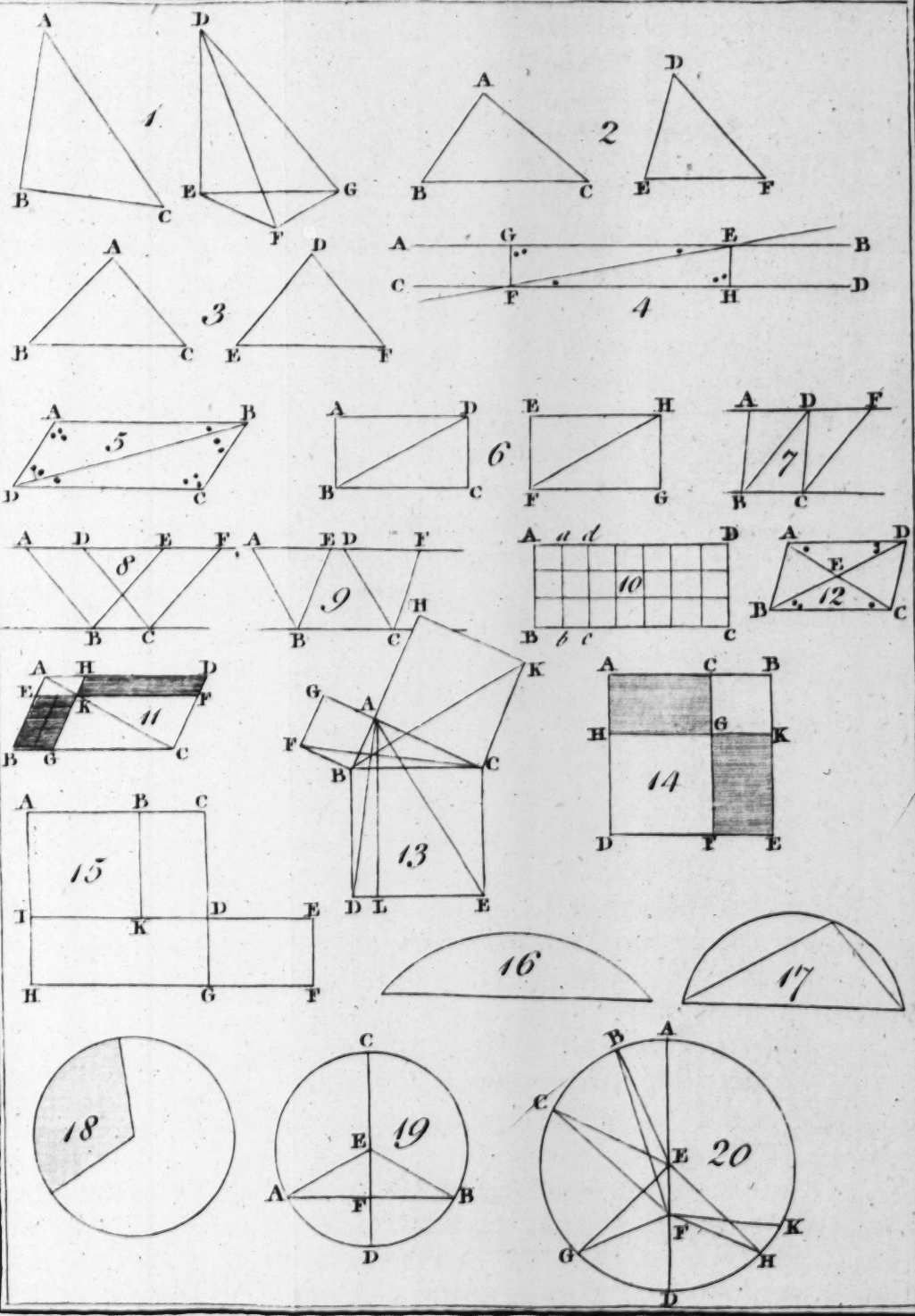
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Γ DB, and DB Γ DA; but they are also equal by the Supposition, which is impossible: $\therefore$ E is not the Center of the Circle ABC, and in the same Manner it may be demonstrated that no other Point but D is the Center. Q. E. D.

137. THEOREM 5. *One Circle cannot cut another in more than two Points.*

For if it be possible, let the Circle ABC cut the Circle DEF in more than two Points, viz. in B, G, F; and let K be the Center of the Circle ABC, and BF, GK, FK, be joined; then, because within the Circle DEF there is taken the Point K, from which to the Circumference DEF fall more than two equal Lines, KB, KG, KF, the Point K must be the Center of the Circle DEF; but it is also the Center of the Circle ABC, and so the two Circles ABC, DEF, would have the same common Center and Radii, and consequently cannot cut each other, but would coincide, and $\therefore$ properly speaking, would be but one Circle: $\therefore$ one Circle cannot cut another in three Points. Q. E. D.

138. A LEMMA, OR THEOREM 6. *If two right-angled Triangles have two Sides of one, AC, CB, equal to two Sides of the other, DF, FE, each to each (viz. $AC = DF$, $CB = FE$;) the remaining Side of one AB, will be = the remaining Side of the other DE.*

For $AB^2 + BC^2 = AC^2$, and $DE^2 + EF^2 = DF^2$: But $AC = DF$ by the Supposition, $\therefore AC^2 = DF^2$, $\therefore AB^2 + BC^2 = DE^2 + EF^2$; but $BC = EF$ by the Supposition, and so $BC^2 = EF^2$, $\therefore$ taking away these equals, there remains $AB^2 = DE^2$, and consequently $AB = DE$. Q. E. D.—
This Demonstration proves the Theorem to be true, when the Side opposite to the Rt. $\angle$ and one of the other Sides of one, are = the Side opposite to the Rt.

$\angle$ and one of the other Sides of the other Δ . But it is equally true when the two Sides containing the Rt. $\angle$ in one = the two Sides containing the Rt. $\angle$ in the other ; for it then becomes the same as Article 58.

Pl. III. *138. THEOREM 7. *Equal Lines, AB, CD, in the Circle ABDC, are equally distant from the Center E; and those AB, CD, equally distant from the Center, are equal to each other.*

First, that if $AB = CD$, they are equidistant from the Center, may be thus demonstrated. Let $EF \perp$ to AB , $EG \perp$ to CD , and join AE and CE . Then the $\angle$ s AFE and CGE being Rt. $\angle$ s, the Δ s AFE , CGE , are ^a Rt. $\angle$ Δ s ; and AB , CD , are ^b bisected by the Lines EF , EG ; and $\therefore$ as their Wholes are equal, their Halves must be equal also, that is, $AF = CG$. Hence, in the two Rt. $\angle$ Δ s AFE , CGE , are two Sides of one AE , AF , respectively = two Sides of the other CE , CG , (for AE , CE , are Radii of the same Circle and $\therefore$ equal,) the remaining Side FE of one is ^c = the remaining Side GE of the other ; which is all that is meant by saying, the Lines AB , CD , are equidistant from the Center. Q. E. D.

Secondly. If AB , CD , are equidistant from the Center, it may be easily demonstrated that they are = each other. For AE being = CE , and $EF = EG$, by the Supposition, the Rt. $\angle$ Δ s AFE , CGE , have two Sides of one AE , EF , = two Sides of the other, CE , EG , and $\therefore AF^c = CG$; but AF , CG , are the ^b Halves of their respective Wholes AB , CD ; and as the Halves AF , CG , are proved to be equal to each other, their Wholes must be so too ; that is, $AB = CD$. Q. E. D.

Pl. 3.F.6. 139. THEOREM 8. *The Diameter AD is the greatest Line in a Circle ABCD ; and of any others BC, FG, that*

that BC which is nearer to the Center E, is always greater than that more remote FG.

Let EI, EH, be the Distances of BC, FG, from the Center E; and let BE, CE, FE, GE, be joined; then, it is manifest that AE being ^a = BE, ED = EC, AD is = BE + CE; but BE + EC ^b Γ BC, $\therefore$ AD Γ BC. Again, the Δ s BEC, FEG, have two Sides BE, EC, of one = two Sides FE, GE, of the other; but the contained $\angle$ of one BEC Γ the contained $\angle$ of the other FEC, $\therefore$ BC ^c Γ FG. Hence we have proved that AB Γ BC, and BC Γ FG; consequently Γ any other Line at the same Distance from the Center (because Lines at the same Distance from the Center are ^d = each other.) Q. E. D. ^a 18. ^b 70. ^c 92. ^d 138.

140. Hence a Line greater than the Diameter, drawn from any Point within the Circle, must cut the Circumference; and $\therefore$ any Line within a Circle, being produced both Ways indefinitely, must cut the Circumference.

141. THEOREM 9. A Line ED meeting the Circle Pl. 3. F. 7. ACG in the Point A, at right Angles to the Radius BA, is a Tangent to the Circle at the Point A. And no Line AK can be drawn from the said Point A between the Line ED and the Circumference, which does not cut the Circle; or, which is the same Thing, no Line AK can make so great an acute Angle BAK with the Diameter CA, or so small an Angle FAK with the Line AD, but that it shall cut the Circle.

First. That ED is a Tangent, may be thus demonstrated. Let any Line BF be drawn; then the Δ BAF being a Rt. $\angle$ Δ , BF^2 ^a = $BA^2 + AF^2$, $\therefore$ BF^2 Γ BA^2 $\therefore$ BF Γ BA the Radius, wherever the Point F is taken in the Line ED, and $\therefore$ the Point F will fall without the Circle, as BF has been just shewn to be Γ the Radius BA, and consequently Γ its ^b equal the Radius BL. $\therefore$ ED ^b 18. touches

touches the Circle in no Point but A, and $\therefore$ it is
 125. a^c Tangent to the Circle at the Point A. Q. E. D.

2dly. If possible, let the Line AK be drawn from A, between the Line ED and the Circumference; and let H be a Point taken in AK, so that BH may be at Rt. $\angle$ s to AK, and let G be the Point where the Line GH meets the Circumference. Then, because BHA is a Rt. $\angle$ Δ , Rt. $\angle$ d at H, the $\angle$ BHA^a = $\angle$ BAH + $\angle$ ABH, the $\angle$ BHA Γ BAH, and $\therefore$ Side BA^b Γ Side BH; but BA^c = BG, $\therefore$ BG Γ BH, the less Γ the greater, which is absurd; $\therefore$ no Line AK can be drawn from the Point A, between ED and the Circumference, which does not cut the Circle; or, in other Words, "no Line AK can make so great an acute $\angle$ BAK with the Diameter CA, or so small an Angle FAK with the Line AD, but that the Circumference shall pass between the Tangent AD and Line AK." Q. E. D. And this is all that *Euclid* must be understood to mean when he says, (in the *Greek* Text, and also in most Translations, that) the $\angle$ of the Semicircle is Γ any rectilineal $\angle$, and the remaining $\angle$ Δ any rectilineal $\angle$.

142. COROLLARY 1. Hence if a Line touches a Circle in any Point, a right Line drawn from the Center to that Point will be perpendicular to the touching Line.

For if that Line be not perpendicular to the Radius at the Point of Contact, it will cut the Circle by this Proposition.

143. COROLL. 2. If a Line touches a Circle, and from the Point of Contact a Line be drawn perpendicular to the touching Line, the Center of the Circle shall be in that Line.

144. *Scholium.* From the Manner in which this Theorem is worded in several Copies of *Euclid*, many strange Paradoxes and Absurdities have been deduced by some modern Mathematicians, which would not have been had they rightly considered what *Euclid* must mean, and not what the Words seem to import.

145. THEOREM 10. *The Angle BEC at the Center of a Circle, is double of the Angle BAC at the Circumference, upon the same Base; that is, upon the same Part of the Circumference BC.* Pl. III. Fig. 8 & 9.

Because $EA = EB$, the Angle $EAB = \angle EBA$, $\therefore \angle EAB + \angle EBA = 2\angle FAB$; but $\angle FEB = \angle EAB + \angle EBA$, $\therefore \angle FEB = 2\angle FAB$. After the same Manner may be shewn that the $\angle FEC = 2\angle FAC$; hence, adding in Figure 8th, we have $\angle FEB + \angle FEC = 2\angle FAB + 2\angle FAC$, that is, $\angle BEC = 2\angle BAC$. Q. E. D. Or subtracting in Figure 9th, we find $\angle FEC - \angle FEB = 2\angle FAC - 2\angle FAB$, that is, $\angle FAC - \angle FAB \times 2 = 2\angle BAC$. Q. E. D.

146. THEOREM 11. *The Angles in the same Segment of a Circle are equal to one another.*

First then, let $ABCDE$ be a Circle, and F the Center; then, if the Segment $BAED$ be a Semicircle, as in Figure 10th. the $\angle$ s BAD , BED , BFD , stand upon the same Base BD , $\therefore$ the $\angle BAD = \frac{1}{2} \angle BFD$, and $\angle BED = \frac{1}{2} \angle BFD$; $\therefore \angle BAD = \angle BED$, wherever the Points A , E , are taken in the Circumference $BAED$. Q. E. D.

Secondly. But if the Segment $BAED$ be a Semicircle, let F be the Center, and suppose Lines drawn as represented in the Figure. Then the Segment

ment BAEC being Γ a Semicircle, the $\angle$ s in it upon the same Base BC are by the first Case equal to each other, that is, $\angle BAC = \angle BEC$; and by the same Reason the $\angle$ s in the Segment CAED on the same Base CD are = each other, viz. $\angle CAD = \angle CED$; $\therefore$ by adding equals, $\angle BAC + \angle CAD$ ^c 46. $= \angle BEC + \angle CED$, that is, $\angle BAD = \angle BED$. Q. E. D.

Pl. III. 147. THEOREM 12. *The opposite Angles of any quadrilateral Figure, ABCD, described in a Circle, are together equal to two right Angles.*

^a 146. Join AC, BD; then the $\angle CAB$ ^a $= \angle CDB$, being in the same Segment BADC, and upon the same Base BC; and the $\angle ACB = \angle ADB$, being in the same Segment ADCB, and upon the same Base AB. $\therefore$ adding equals, the $\angle ADB + \angle CDB$ ^b 46. $= \angle ACB + \angle CAB$, that is, $\angle ADC = \angle ACB + \angle CAB$; to each of these equals adding the $\angle ABC$, we have $\angle ADC + \angle ABC = \angle ACB + \angle CAB + \angle ABC$; but the $\angle ACB + \angle CAB + \angle ABC$ ^c 78. $= 2$ Rt. $\angle$ s, $\therefore \angle ADC + \angle ABC$ ^d 45. $= 2$ Rt. $\angle$ s. In the same Manner it may be shewn that the $\angle BAD + \angle BCD = 2$ Rt. $\angle$ s. Q. E. D.

Pl. III. 148. THEOREM 13. *The Angle in a Semicircle BAC is a right Angle; but the Angle in a Segment ABC, greater than a Semicircle, is less than a right Angle; and the Angle in a Segment ADC, less than a Semicircle, is greater than a right Angle.*

Let E be the Center, and Lines be drawn as in the Figure. Then, AE being ^a 18. $= EB$, the $\angle EAB$ ^b 59. $= \angle EBA$; and EA being $= EC$, the $\angle EAC = \angle ECA$. $\therefore$ adding equals, $\angle EAB + \angle EAC$ ^c 46. $= \angle EBA + \angle ECA$, that is, $\angle BAC = \angle CBA + \angle BCA$; but, $\angle FAC$ ^d 76. $= \angle CBA + \angle BCA$, $\therefore \angle BAC$ ^e 45. $= \angle FAC$, $\therefore$ each of them

is a ^f right $\angle$, $\therefore$ the $\angle BAC$ in a Semicircle is a Rt. $\angle$. Q. E. D. f 14.

2dly. The $\triangle BAC$ having been just shewn to be Rt. $\angle$ d. viz. the $\angle BAC$ a Right $\angle$, the $\angle ABC$ (in the Segment ABC) must be ^a $\wedge$ a Rt. $\angle$. e 86.
Q. E. D.

3dly. The Figure $BADC$ being quadrilateral, the $\angle ABC + \angle ADC^h = 2$ Rt. $\angle$ s; but it has been h 147.
just shewn that $\angle ABC \wedge$ a Rt. $\angle$, consequently $\angle ADC$ must be Γ a Rt. $\angle$. Q. E. D.

149. THEOREM 14. In an oblique Triangle ACD , Pl. III. inscribed in a Circle, the vertical Angle ADC exceeds, F. 14. & 15.
or is less than a right Angle, by the Angle CAB , made by the Base AC and the Diameter AB , drawn from either Extremity of the Base.

Join DB ; then, AB being a Diameter, $ADCB$ ^a 22.
is ^a a Semicircle, and $\therefore \angle ADB^b =$ a Rt. $\angle$, and ^b 148.
the $\angle CDB^c = \angle CAB$, (being in the same Seg- ^c 146.
ment $CDAB$, and standing on the same Base BC ;) $\therefore$
adding equals in Fig. 14. the $\angle ADB + \angle CDB^d$ ^d 46.
 $=$ a Rt. $\angle + \angle CAB$; but $\angle ADB + \angle CDB =$
the whole $\angle ADC$, $\therefore \angle ADC =$ a Rt. $\angle + \angle$
 CAB . Q. E. D. And in Fig. 15. subtracting
equals, the $\angle ADB - \angle CDB^e =$ a Rt. $\angle - \angle$ ^e 47.
 CAB ; but $\angle ADB - \angle CDB = \angle ADC$, $\therefore \angle$
 $ADC =$ a Rt. $\angle - \angle CAB$. Q. E. D.

150. THEOREM 15. In equal Circles, viz. Circles Pl. III. which have equal Radii, the Angles A, G , at the Cir- F. 16 & 17.
cumferences, and standing upon equal Chords BC, EF ,
are equal to each other. And if the Angles at the Cir-
cumferences are equal, the Chords on which they stand
will be equal also.

Let D and H be the Centers, and let Lines be
drawn as in the Figures. 1. Then, BC being $=$
 G EF

EF by the Supposition, and $BD = CD = EH = FH$ being equal Radii, the Δ s BDC, EHF, have three Sides of one respectively = three Sides of the other, and $\therefore \angle BDC^a = \angle EHF$; but the $\angle$ s A, and G, are b Halves of the $\angle$ s BDC, EHF; and as the Wholes are equal to each other, 'tis manifest their Halves must be so too, that is, $\angle BAC = \angle G$. Q. E. D.

2dly. If the $\angle A = \angle G$; as the $\angle$ s A and G, are the Halves b of the $\angle$ s BDC, CHF, respectively, and as the Halves being equal, the Wholes must be so too, the $\angle BDC$ must be $= \angle EHF$. But $BD = EH$, $DC = HF$, being equal Radii, the Δ s BDC, EHF, have two Sides BD, CD, and $\angle$ contained BDC of one = two Sides EH, FH, and $\angle$ contained EHF of the other; and consequently the Δ s themselves a equal in all Respects; $\therefore$ the Base BC = Base EF. Q. E. D.

Pl. IV.
Fig. 1.

151. THEOREM 16. *Let AB be the Diameter of the Circle ADCB, and EF a Line perpendicular thereto, touching the Circle in the Point B; and BD a Line cutting the Circle; then I say, the Angles made by the touching Line EF and cutting Line BD, shall be equal to the Angles in the alternate Segments; viz. $\angle DBF = \angle DAB$, and $\angle DBE = \angle DCB$, wherever the Points A and C are taken in the Segments, DAB, DCB.*

For the vertical $\angle DAB$ of the ΔDAB is $\wedge$ a Rt. $\angle$ by the $^a \angle DBA$, made by the Base DB and Diameter BA; and $\therefore$ as ABF is a Rt. $\angle$, must be $= \angle DBF$. Q. E. D.

Again, the vertical $\angle DCB$ of the ΔDCB exceeds a Rt. $\angle$ by the said $\angle DBA$, and $\therefore$ as ABE is a Rt. $\angle$, the $\angle DCB$ must be $= (\angle ABE + \angle ABD) = \angle DBE$. Q. E. D. And the $\angle DAB$ continues the b same wherever the Point A is taken in the

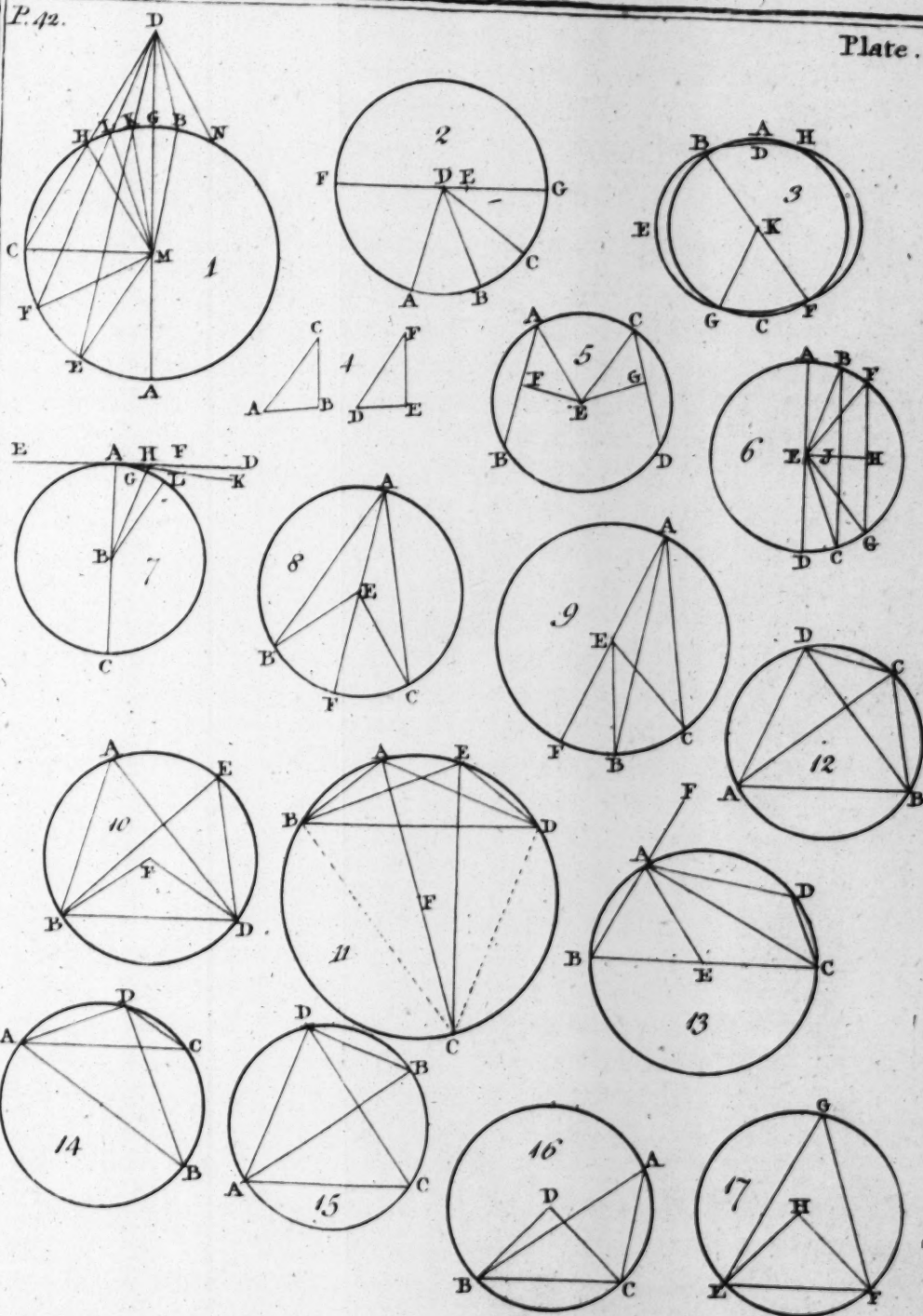
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the Circumference DAB, as does the $\angle$ DCB where-
 ever the Point C is taken in the Circumference DCB.
 $\therefore$ this Theorem is true.

B O O K IV.

Of PROPORTION, containing what is most valuable
 in Euclid's 5th and 6th Books.

152. Def. 1. **A** Lesser Magnitude is said to be an
aliquot Part of a greater Magni-
 tude, when the lesser measures the greater, *i. e.*
 when the lesser is contained a certain Number of
 Times exactly in the greater. Thus, a is an aliquot
 Part of $3a$, a being contained in $3a$, 3 Times.

153. Def. 2. A greater Magnitude is said to be a
Multiple of a lesser, when the lesser measures the
 greater, *i. e.* when the lesser is contained in the
 greater, exactly a certain Number of Times. Thus,
 $3a$ is a Multiple of a , a being contained in $3a$, 3
 Times, and $n \times b$ is a Multiple of b , for b is con-
 tained n Times in $n \times b$.

154. The 3d Definition, "That Ratio is a mu-
 tual Relation of two Magnitudes of the same Kind
 to one another in Respect of Quantity," is rather
 metaphysical than mathematical, and has been the
 Cause of much Controversy amongst the modern
 Mathematicians. Indeed it might well be omitted
 without any Loss, and, as Mr. *Simson* justly ob-
 serves, seems to be the Addition of some unskilful
 Editor.

155. Def. 4. These Magnitudes are said to have a
 Ratio to one another, the lesser of which can be
 multiplied so as to exceed the other.

156. *Def. 5.* The first of four Magnitudes is said to have the same Ratio to the second which the third has to the fourth, when any Equimultiples whatsoever of the first and third being taken, and any Equimultiples whatsoever of the second and fourth : If the Multiple of the first be less than that of the second, the Multiple of the third is also less than that of the fourth ; or, if the Multiple of the first be equal to that of the second, the Multiple of the third is also equal to that of the fourth ; or, if the Multiple of the first be greater than that of the second, the Multiple of the third is also greater than that of the fourth.*

157. *Def. 6.* Magnitudes which have the same Ratio are called Proportionals.

158. *Def. 7.* Of a greater and less Ratio being not used in this Essay, we have thought proper to omit it, as the Theorems which are obscurely, (if not inaccurately) demonstrated by it, may be demonstrated more clearly by other Means.

159. The same may be said of the 8th Definition, viz. " That Analogy or Proportion is a Similitude of Ratios," as was remarked concerning the 3d. Definition.

160. *Def. 9.* Proportion consists in three Terms at least.

161.

* This is the true geometrical Definition of the Ancients ; however it may not be improper to give also the numerical or algebraical Definition, viz. Four Quantities are said to be in direct Proportion to each other, when the Quotient of the first by the second is equal to the Quotient of the third by the fourth : Which Quotients are called Ratios. Or in other Words, Ratio is the Number expressing how often the Consequent is contained in the Antecedent ; or by which the Consequent being multiplied, the Product will be equal to the Antecedent.

161. *Def. 10.* When three Magnitudes are Proportionals, the first is said to have to the third the duplicate Ratio of that which it has to the second.

162. *Def. 11.* When four Magnitudes are continual Proportionals, the first is said to have to the fourth the triplicate Ratio of that which it has to the second, and so forward quadruplicate, &c. increasing the Denomination each Time by Unity, in any Number of Proportionals.

163. *Def. 12, of Compound Ratio.* When there are any Number of Magnitudes of the same Kind, the first is said to have to the last of them the Ratio compounded of the Ratio which the first has to the second, and of the Ratio which the second has to the third, and of the Ratio which the third has to the fourth, and so forward unto the last Magnitude.

For Example, if A, B, C, D, be four Magnitudes of the same Kind, the first A is said to have to the last D the Ratio compounded of the Ratio of A to B, and of the Ratio of B to C, and of the Ratio of C to D; or, the Ratio of A to D is said to be compounded of the Ratios of A to B, B to C, and C to D.

And if A has to B the same Ratio which E has to F; and B to C the same Ratio which G has to H; and C to D the same Ratio that K has to L; then, by this Definition, A is said to have to D the Ratio compounded of Ratios which are the same with the Ratios of E to F, G to H, and K to L. And the same Thing is to be understood when it is more briefly expressed by saying A has to D the Ratio compounded of the Ratios of E to F, G to H, and K to L.

In like Manner, the same Things being supposed, if M has to N the same Ratio which A has to D, then, for Shortness Sake, M is said to have to N the Ratio

Ratio compounded of the Ratios of E to F, G to H, and K to L.

164. *Scholium.* Nothing has been thought more difficult, or caused greater Disputes in Geometry, than the Definition of compound Ratio given by Theon in the 5th Def. of the 6th Book of Euclid, viz. "A Ratio is said to be compounded of Ratios when the Quantities of Ratios being multiplied by one another, makes a certain Ratio;" which is faulty, for there can be no Multiplication but by a Number, and so the Definition will be ungeometrical. As to the Use of compound Ratio, it is introduced into Geometry only to render some Demonstrations more concise, or that some Propositions may be more compendiously enounced. It may be further observed, "that in any Magnitudes whatever, of the same Kind, A, B, C, D, &c. the Ratio compounded of the Ratios of the first to the second, of the second to the third, and so on to the last, is only a Name or Expression, by which the Ratio which the first A has to the last D is signified, and by which at the same Time the Ratios of all the Magnitudes, A to B, B to C, C to D, from the first to the last, to one another, whether they be the same or be not the same, are indicated; as in Magnitudes which are continual Proportionals, A, B, C, D, &c. the duplicate Ratio of the first to the second is only a Name, or Expression, by which the Ratio of the first A to the third C is signified, and by which, at the same Time, is shewn that there are two Ratios of the Magnitudes from the first to the last, viz. of the first A to the second B, and of the second B to the third or last C, which are the same to one another; and the triplicate Ratio of the first to the second is a Name, or Expression, by which the Ratio of the first A to the fourth D is signified, and by which, at the same Time, is shewn that there are three Ratios of the Magnitudes from the first

“ first to the last, *viz.* of the first A to the second
 “ B, and of B to the third C, and of C to the fourth
 “ or last D, which are all the same to one another;
 “ and so in the Case of any other multiplicate Ra-
 “ tios.” And this is the right Meaning of these
 Ratios as used by *Euclid*, *Archimedes*, *Apollonius*,
 and other ancient Geometricians. However, that
 of *Theon* abovementioned is still retained by most
 Moderns, and indeed may be made Use of when we
 are discoursing of Numbers, or of Magnitudes as
 expressed by Numbers, though it would be improper
 in this on Geometry.

165. Def. 13. In Proportionals the antecedent
 Terms are called homologous to one another, as
 also the Consequents to one another.

Note. Geometricians make Use of the following
 technical Words to signify certain Ways of changing
 either the Order or Magnitude of Proportionals, so as
 to continue still to be Proportionals.

166. Def. 14. *Permutando*, or *alternando*, by
Permutation, or *alternately*: This Word is used
 when there are four Proportionals, and it is inferred,
 that the first has the same Ratio to the third which
 the second has to the fourth; or that the first is
 unto the third, as the second to the fourth; as is
 shewn in Article 199.

167. Def. 15. *Invertendo*, by *Inversion*; when
 there are four Proportionals, and it is inferred, that
 the second is unto the first as the fourth to the third.
 See Art. 198.

168. Def. 16. *Componendo*, by *Composition*; when
 there are four Proportionals, and it is inferred,
 that the first, together with the second, is to the se-
 cond, as the third, together with the fourth, is to
 the fourth. See Art. 200.

169. Def. 17. *Dividendo*, by *Division*; when there are four Proportionals, and it is inferred, that the Excess of the first above the second is to the second, as the Excess of the third above the fourth is to the fourth. See Art. 201.

170. Def. 18. *Convertendo*, by *Conversion*; when there are four Proportionals, and it is inferred, that the first is to its Excess above the second, as the third to its Excess above the fourth. See Art. 202.

171. Def. 19. *Ex aequali*, (scil. *Distantia*,) or *ex aequo*, from *Equality of Distance*; when there is any Number of Magnitudes more than two, and as many others so that they are Proportionals when taken two and two of each Rank, and it is inferred, that the first is to the last of the first Rank of Magnitudes, as the first is to the last of the others. — Of this there are the two following Kinds, which arise from the different Order in which the Magnitudes are taken two and two.

172. Def. 20. *Ex aequali*, from *Equality*: This Term is used simply by itself, when the first Magnitude is to the second of the first Rank, as the first to the second of the other Rank; and as the second is to the third of the first Rank, so is the second to the third of the other; and so forward in Order, and the Inference is as mentioned in the preceding Definition; whence this is called *Ordinate Proportion*.

173. Def. 21. *Ex aequali*, in *proportione perturbata*, seu *inordinata*, from *Equality*, in *perturbate* or *disorderly Proportion*: This Term is used when the first Magnitude is to the second of the first Rank, as the last but one is to the last of the second Rank; and as the second is to the third of the first Rank, so is the last but two to the last but one of the second Rank; and as the third is to the fourth of the first Rank,

Rank,

Rank, so is the third from the last to the last but two of the second Rank, and so forward in a cross Order; and the Inference is as in the 19th Definition.

174. *Def. 22.* Similar rectilineal Figures are those which have their several Angles equal, each to each, and the Sides about the equal Angles Proportionals.

175. *Def. 23.* Two Magnitudes are said to be reciprocally proportional to two others, when one of the first is to one of the other Magnitudes, as the remaining one of the last two is to the remaining one of the first.

176. *Def. 24.* A Line is said to be cut in *extreme and mean Ratio*, when the Whole is to the greater Segment as the greater Segment is to the less.

177. *Axiom 1.* Equimultiples of the same, or equal Magnitudes, are equal to one another; that is, if $A = B$, then any Number of Times $A =$ same Number of Times B ; so $mA = mB$.

178. *Axiom 2.* These Magnitudes of which the same, or equal Magnitudes, are Equimultiples, are equal to one another.

179. *Axiom 3.* A Multiple of a greater Magnitude is greater than the same Multiple of a lesser.

180. *Axiom 4.* That Magnitude whose Multiple is greater than the same Multiple of another, is greater than that other Magnitude.

181. *Axiom 5.* Magnitudes having the same Ratio to one and the same or equal Magnitudes, are equal to each other.

182. *Axiom 6.* Equal Magnitudes have to one and the same Magnitude the same Ratio : And the same unto equal Magnitudes has the same Ratio.

183. *Axiom 7.* Magnitudes to which one and the same Magnitude has the same Ratio, are equal.

184. *Axiom 8.* If two Magnitudes be referred to a third, that is the greatest which has the greatest Ratio.

185. *Axiom 9.* Ratios equal to one and the same Ratio, or to equal Ratios, are also equal to one another.

186. *Axiom 10.* Unequal Magnitudes, compared with one and the same or equal Magnitudes, cannot have equal Ratios.

187. **THEOREM I.** *Magnitudes have the same Ratio to one another which their Equimultiples have. Let a and b represent the two Magnitudes, and let m be the Number of Times each of these are contained in its Multiple ; then, ma, mb, represent the Equimultiples, and what we affirm is, that $ma : mb :: a : b$.**

For taking any Equimultiples of the first and third, viz. cma , ca , and any Equimultiples, dmb , db , of the second and fourth ; it is plain that if $cma = dmb$, as they are Equimultiples of ca , db , that $ca^* = db$; and if $cma \wedge dmb$, then $ca \dagger \wedge db$; also if

* 178.
† 180.

* It may be demonstrated algebraically thus. The Quotient of ma by mb is $\frac{ma}{mb} = \frac{a}{b}$; and the Quotient of a divided by b is also $\frac{a}{b}$. $\therefore ma : mb :: a : b$. Q. E. D.

if $ema \propto dmb$, then $ca \propto db$. $\therefore$ by Art. 156, as
 $ma : mb :: a : b$. Q. E. D.

188. THEOREM 2. If four Quantities, a, b, c, d , are proportional, viz. as $a : b :: c : d$; then any Equimultiples of the two first will have the same Ratio as any Equimultiples of the two last, viz. as
 $ma : mb :: nc : nd$.

For by Art. 187, As, $ma : mb :: a : b$,

And, as, $nc : nd :: c : d$.

But a is to b , in same Ratio as c to d by the Supposition, $\therefore$ by Art. 185, as $ma : mb :: nc : nd$.
 Q. E. D.

189. THEOREM 3. Triangles ABC, ACD, and Pl. IV.
 Parallelograms EC, CF, of the same Altitude, are to F. 2.
 one another as their Bases, BC, CD.*

Let BD be produced both Ways to the Points H, L, and let BG, GH, be each = BC; and DK, KL, each = CD; and let Lines be drawn as in the Figure. Then, because the Δ s AHG, AGB, ABC, have equal Bases and the same Altitude, they are* = 109. each other; $\therefore$ whatever Multiple the Base HC is of the Base BC, the same Multiple is the Δ AHC of the Δ ABC. By the same Reason, whatever Multiple the Base LC is of the Base CD, the same Multiple

* The algebraic Demonstration may stand thus. Let a = Base of the Parallelogram EC, b = Base of CF, and h = their Height; then ab = * Area of EC, and bh = Area of CF. Now it is plain that as $ab : bh :: a : b$; for the Quotient of the first Term

by the second is $\frac{ab}{bh} = \frac{a}{b}$; and the Quotient of the third by the

4th is also $\frac{a}{b}$; $\therefore$ it will be, as $ab : bh :: a : b$. Q. E. D.

And as Δ s are Halves of the Parallelograms, as the Parallelograms are as their Bases, their Halves, viz. the Δ s, must be in the same Ratio also. Q. E. D.

- * 109. tiple is the $\triangle ALC$ of the $\triangle ADC$. And if the Base HC be $=$ Base CL , the $\triangle AHC$ will be $=$ the $\triangle ALC$; and if the Base HC be $\neq$ CL , the $\triangle AHC \neq \triangle ALC$; and if HC be $\triangle$ CL , the $\triangle AHC$ is $\triangle$ $\triangle ALC$. $\therefore$ ^b as Base BC : Base $CD :: \triangle ABC : \triangle ACD$. Q. E. D.

- * 107. Again, since we have demonstrated that the $\triangle$ s ABC , ACD , are to each other as their Bases BC , CD , and the Parallelograms EC , CF , are ^c double of the $\triangle$ s; it must follow, that as the Halves are in Proportion of their Bases, their Wholes (*viz.* the Parallelograms) must be ^d so too. Q. E. D.

190. COROLLARY. From hence it is plain, that $\triangle$ s and Parallelograms which have equal Altitudes, are to one another as their Bases.

- Pl. IV. F. 3, 4, 5. 191. THEOREM 4. If a Line DC be drawn parallel to one of the Sides, BC , of a $\triangle ABC$, it shall cut the other Sides, or these Sides produced, proportionally, (*viz.* $BD : DA :: CE : EA$, or $AB : AD :: AC : AE$.) And if the Sides, or the Sides produced, be cut proportionally, the Line DE , which joins the Points of Section, shall be parallel to the remaining Side of the Triangle, *viz.* DE parallel to BC .

- * 109. For join BE , CD , then the $\triangle$ s BDE , CDF , being on the same Base DE , and between the same
- * 182. $\parallel$ s DE , BC , are $=$ each other. Hence, as $\triangle BDE : \triangle ADE ::$ ^b $\triangle CDE : \triangle ADE$. And if we consider AD , BD , as the Bases of the $\triangle$ s AED , BED , the Point E may be considered as the Vertex of both, and so a $\perp$ drawn from E to AB would be common to both; that is, both would have the
- * 189. same Altitude; $\therefore$ As $\triangle BDE : \triangle ADE ::$ ^c $BD : DA$. And by the same Reason, as $\triangle CDE : \triangle ADE ::$ ^d $CE : EA$. Q. E. D. And, by a like Method of Reasoning,
- * 185.

soning, it may be shewn that $AB : AD :: AC : AE$. Q. E. D.

The second Part of the Theorem may be thus demonstrated. By the Supposition, as $BD : DA :: CE : EA$; and as $BD : DA :: {}^e \triangle BDE : \triangle ADE$; and as $CE : EA :: \triangle CDE : \triangle ADE$; $\therefore$ As $\triangle BDE : ADE :: {}^d \triangle CDE : \triangle ADE$ $\therefore$ the $\triangle$ s BDE, CDE, have the same Ratio to the $\triangle$ ADE, and $\therefore$ the $\triangle BDE = \triangle CDE$, and stands on the same Base DE: But it is manifest that equal $\triangle$ s must have equal Areas, and $\therefore$ as the Area of a $\triangle$ is $= \frac{1}{2}$ its Base into its Altitude, and they have the same Base, they must consequently have the same Altitude, otherwise their Areas could not be equal; that is, the $\perp$ Distance of the Points D and E, from the Line BC, are equal, and $\therefore DE \parallel$ to BE. Q. E. D.

192. THEOREM 5. *If the Angle BAC, of a Triangle ABC, be divided into two equal Angles by a Line AD, cutting the Base, the Segments of the Base will have the same Ratio that the other two Sides of the Triangle have to one another; viz. as $BD : DC :: AB : AC$. And if the Segments of the Base have the same Ratio which the other Sides of the Triangle have to one another, the Line AD, drawn from the Vertex to the Point of cutting, will divide the $\angle BAC$ of the Triangle into two equal Parts; viz. $\angle BAD = \angle CAD$.*

Pl. IV.
F. 6.

First. Let BA be produced till $AE = AC$, and let CE be joined; then, the $\angle ACE = \angle E$: But the $\angle BAC = \angle ACE + \angle E$, and as these $\angle$ s are equal, each of them must be $= \frac{1}{2} \angle BAC = \angle DAC$, that is, $\angle DAC = \angle ACE$; $\therefore AD, EC$, are $\parallel$ to each other, $\therefore$ As $BD : DC :: BA : AE$. Q. E. D.

Secondly. By the Supposition, $BD : DC :: AB : AC$; but $AC = AE$ by the Construction; $\therefore$ as $BD : DC :: AB : AE$; $\therefore AD$ is $\parallel$ to EC ; $\therefore \angle$

^c 40. $\therefore \angle BAD = \angle E$, and $\angle DAC = \angle ACE =$
^f 72. $\angle E$; $\therefore \angle BAD = \angle DAC$. Q. E. D.
^a 59.
^b 45.

193. THEOREM 6. *If four Lines, or Numbers, denoted by the Letters, a, b, c, d, are proportional, viz. as $a : b :: c : d$; the Rectangle contained by the Extremes is equal to the Rectangle under the Means, viz. $a \times d = b \times c$. And if the Rectangle contained by the Extremes be = the Rectangle contained by the Means, these four Quantities are proportional; that is, if $a \times d = b \times c$, then it will be, as $a : b :: c : d$.*

^a 187. First. Since $a \times d$, and $b \times d$, are Equimultiples of a and b , we have, as $a \times d : b \times d :: a : b$.
 And $\therefore$ since by the Supposition, $a : b :: c : d$,
^b 185. we have $a \times d : b \times d :: c : d$. But $b \times c$, and $b \times d$ are Equimultiples of c and d , $\therefore b \times c : b \times d :: c : d$;
^c 187. $\therefore a \times d : b \times d :: b \times c : b \times d$;
^d 185. $\therefore a \times d = b \times c$. Q. E. D.
^e 181.

^a 187. Secondly. It is plain, that $a \times d$, $b \times d$, are Equimultiples of a and b , $\therefore a : b :: a \times d : b \times d$; and $b \times c$, $b \times d$, are Equimultiples of c , and d ; $\therefore c : d :: b \times c : b \times d$; but $a \times d = b \times c$ by the Supposition, $\therefore$ by writing $a \times d$, for its equal $b \times c$, we have, as $c : d :: a \times d : b \times d$; but it has been just shewn that $a : b :: a \times d : b \times d$, $\therefore c : d$ has the same Ratio as $a : b$, that is, as $a : b :: c : d$. Q. E. D.

194.

* This Theorem may be demonstrated algebraically thus. First,
^a 156. $\frac{a}{b} = \frac{c}{d} \therefore$ multiplying by b we have $a = \frac{bc}{d}$, and this by d gives $da = bc$. Q. E. D. Secondly. If we divide each Side of the Equation $da = bc$ by d , we have $a = \frac{bc}{d}$, and this again by b gives $\frac{a}{b} = \frac{c}{d}$; $\therefore$ by Art. 156, As $a : b :: c : d$. Q. E. D.

194. THEOREM 7. *Equiangular Triangles are similar; that is, the Sides about the equal Angles are Proportionals.*

Let $Da = AB$, $Db = AC$, then the $\angle D$ being $=$ Pl. IV.
 $\angle A$ by the Supposition, the Δ s Dab , ABC , are $=$ F. 7.
 equal in every Respect; $\therefore \angle Dab = \angle ABC$; but $=$ 58.
 $\angle ABC = \angle DEF$ by the Supposition, $\therefore \angle Dab =$ $=$ 45.
 $\angle DEF$, $\therefore ba^c \parallel$ to EF ; $\therefore DE : Da ::$ * $DF :$
 Db . $\therefore$ for Da , Db , writing their equals AB , AC ,
 the Proportion becomes $DE : AB :: DF : AC$.
 Q. E. D. $=$ 40.
 * 191.

And if Fd be taken $= CA$, and $Fc = CB$, it may be shewn in the same Manner, that $FD : CA :: FE : CB$. Q. E. D.

By joining these two Proportions in one, we have $DE : AB :: DF : AC :: FE : CB$. Q. E. D.

195. THEOREM 8. *If the outward Angle CAE, of a Triangle ABC, is bisected by a Line AD, cutting the Base BC produced in D; then the Segments BD, DC, will have the same Ratio as the other Sides of the Triangle; viz. $BD : DC :: BA : AC$. And if $BD : DC :: BA : AC$, the Line AD will bisect the outward Angle CAE, viz. $\angle CAD = \angle DAE$.* Pl. IV.
 F. 8.

For let CB be $\parallel$ to DA ; then $\angle CFA^a = \angle DAE$, and also $\angle ACF^b = \angle DAC$; but the $\angle$ $=$ 40.
 $DAE = \angle DAC$ by the Supposition, $\therefore \angle CFA =$ $=$ 72.
 $\angle ACF$, $\therefore AF^d = AC$. Now, as CF is $\parallel$ to $=$ 45.
 AD by the Supposition, it follows, $BD : DC^e ::$ $=$ 62.
 $BA : AF$; but it has been just shewn that $AF =$ $=$ 191.
 AC , $\therefore$ for AF writing its equal AC , the above
 Proportion becomes $BD : DC :: BA : AC$.
 Q. E. D.

Secondly. By the Supposition, $BD : DC :: BA : AC$, and FC being $\parallel$ to AD , $BD : DC :: BA :$ $=$ 185.
 AF , $\therefore BA : AC^f :: BA : AF$; $\therefore AC^g = AF$, $=$ 181.
 and $\therefore \angle AFC^h = \angle ACF$; also the $\angle AFC^i =$ $=$ 59.
 DAE , $=$ 40.

^b 72. DAE, and $\angle ACF^b = \angle CAD$, $\therefore \angle DAE (= ^c$
^c 45. $ACF)^c = CAD$. Q. E. D.

Pl. IV. 196. THEOREM 9. *If two Triangles, ABC, DEF,*
 F. 9. *have one Angle A, of one, equal to one Angle D, of the*
other, and the Sides containing the equal Angles propor-
tional, viz. $AB : AC :: DE : DF$, the Triangles
are equiangular and similar.

Let $Af = Df$, and $Ae = DE$; and let fe be
 joined. Then the $\angle A$ being $=$ the $\angle D$ by the
^a 58. Supposition, the Δ s Afe , DFE , are ^a equal in
 every Respect. By the Supposition, $AB : AC ::$
 $DE : DF$; or, which is the same, by writing
 equals, (*viz.* Ae for DE , and Af for DF ;) $AB :$
^b 191. $AC :: Ae : Af$. $\therefore fe$ is ^b $\parallel$ to BC , $\therefore \angle Afe^c =$
^c 40. $\angle ACB$, and $\angle Aef = \angle ABC$; $\therefore$ the Δ s Afe ,
 ABC , are equiangular, and consequently, as the Δ s
 Afe , DFE , have been just shewn to be equal in
 every Respect, ABC , DEF , are also equiangular;
^a 194. and $\therefore$ also ^a similar. Q. E. D.

Pl. IV. 197. THEOREM 10. *If two Triangles, ABC,*
 F. 9. DEF , *have their like Sides proportional, (thus, $AB :$*
 $AC :: DE : DF$; *and $AB : BC :: DE : FE$.)*
the Triangles are similar to each other.

Let $Af = DF$, and $Ae = DE$, and let fe be
 joined; then by the Supposition, $AB : AC :: DE$
 $: DF$; and by writing equals, *viz.* Ae for DE ,
 and Af for DF , the Analogy is, $AB : AC :: Ae :$
^a 191. Af ; $\therefore fe$ is ^a $\parallel$ to BC . Again, by the Supposi-
 tion, $AB : BC :: DE : FE$; and the Δ s ABC ,
^b 196. Aef . being ^b equiangular and similar, $AB : BC ::$
 $Ae : fe$; $\therefore$ by Equality of Ratios, $DE : FE^c ::$
^c 185. $Ae : fe$; but $Ae = DE$ by the Construction, $\therefore$
 writing DE for its equal Ae , we have, $DE : FE ::$
 $DE : fe$, consequently $FE = fe$. Hence the three
 Sides of the Δ s Afe , DFE , have been proved to be
 respectively

respectively equal to each other, and $\therefore$ the Δ s Afe, DFE, are ^d equal in all Respects; and consequently, ^d 95^a as the Δ Afe is similar to the Δ ABC, the Δ DEF must be so too. Q. E. D.

198. THEOREM 11. *If four Quantities are proportional, they will also be Proportionals when taken inversely; viz. if $a : b :: c : d$, then also, $b : a :: d : c$.*

For the Rectangle of the Extremes being $\ast =$ the $\ast$ 193^d Rectangle of the Means, $a \times d = b \times c$; or, which is the same, $b \times c = a \times d$, $\therefore b : a \ast :: d : c$. Q. E. D.

199. THEOREM 12. *If four Quantities be proportional, they will also be proportional when taken alternately; that is, if $a : b :: c : d$, then $a : c :: b : d$.*

For the Rectangle of the Extremes being $\ast =$ Rectangle of the Means, $a \times d = b \times c$, or, which is $\ast$ 193^d the same, $a \times d = c \times b$, $\therefore a : c \ast :: b : d$. Q. E. D.

200. THEOREM 13. *If four Lines, denoted by a , b , c , d , are Proportionals, viz. $a : b :: c : d$; they will also be compoundedly proportional; that is, $a + b : b :: c + d : d$.†*

Let $AB = a$, $BC = b$, $AE = c$, $ED = d$; Pl. IV: then is $AC = a + b$, and $AD = c + d$. Because Fig. 10^a
I by

† Algebraically demonstrated thus. For the Product of the Extremes, $\overline{a+b} \times d = ad + bd$, and the Product of the Means, $b \times \overline{c+d} = bc + bd$; but $a : b :: c : d$, by the Supposition; $\therefore ad = bc$, consequently $ad + bd = bc + bd$, or, which is the same, $\overline{a+b} \times d = b \times \overline{c+d}$, and $\therefore$ by Art. 193. $a + b : b :: c + d : d$. Q. E. D.

- * 191. by the Supposition, $a : b :: c : d$, that is, $AB : BC :: AE : ED$; BE, CD , are * || to each other, and $\therefore AC : BC :: AD : DE$, that is, $a + b : b :: c + d : d$. Q. E. D.

201. THEOREM 14. *If four Lines, denoted by a, b, c, d , are Proportionals, viz. $a : b :: c : d$, they will also be Proportionals by Division, viz. $a - b : b :: c - d : d$.†*

- PL. IV. Let $AC = a, CB = b, AD = c, DE = d$;
Fig. 11. then, $AB = a - b, AE = c - d$; by the Supposition, $a : b :: c : d$, that is, $AC : CB :: AD : DE$, $\therefore BD, DC$, are * || to each other, and $\therefore AB : BC :: AE : ED$; that is, $a - b : b :: c - d : d$. Q. E. D.

202. THEOREM 15: *If four Lines, denoted by a, b, c, d , are Proportionals, viz. $a : b :: c : d$, then they will also be Proportionals by Conversion, that is, $a : a - b :: c : c - d$.**

Let $AC = a, CB = b$, &c. as in the last Theorem; then BC, CD , being there shewn to be || to each

† This may be demonstrated analytically thus. The Product of the Extremes, $\overline{a-b} \times d = ad - bd$, and the Product of the Means $b \times \overline{c-d} = bc - bd$; but $ad = bc$ by Art. 193, consequently $ad - bd = bc - bd$; or, which is the same, $\overline{a-b} \times d = b \times \overline{c-d}$; $\therefore a - b : b :: c - d : d$. Q. E. D.

* Algebraically thus. The Product of the Extremes $a \times \overline{c-d} = ac - ad$, and of the Means $\overline{a-b} \times c = ac - bc$; but it has been before shewn that $ad = bc$; consequently $ac - ad = ac - bc$; and $\therefore$ their Equals are equal to each other, that is, $a \times \overline{c-d} = \overline{a-b} \times c$, or Product of the Extremes = the Product of the Means, and $\therefore$ these Quantities are proportional by Art. 193, as asserted in the Theorem. Q. E. D.

each other, we have $AC : AB :: AD : AE$, that ^{* 191.}
is, $a : a - b :: c : c - d$. Q. E. D.

203. THEOREM 16. *If four Lines, denoted by a, b, c, d, are Proportionals, viz. $a : b :: c : d$, then it will also be, $a + b : a - b :: c + d : c - d$. This is usually cited by the Word mixtly.**

Let $AE = a$, EC and EG each $= b$, $AD = c$, Pl. IV.
 DB and DF each $= d$, then $AC = a + b$, $AG = a - b$, $AB = c + d$, and $AF = c - d$. Fig. 12.
Since by the Supposition, $a : b :: c : d$, that is, $AE : EC :: AD : DB$; and $AE : EG :: AD : DF$; BC and GF are ^{*} $\parallel$ to ED , $\therefore AC : AG :: AB : AF$, that is, $a + b : a - b :: c + d : c - d$. ^{* 191.}
Q. E. D.

204. SCHOLIUM. What has been, or may hereafter be demonstrated, concerning Lines being Proportionals, must, it is manifest, hold good in any other Magnitudes; for Lines may be assumed, having the same Ratio to each other as the Magnitudes themselves; and if the Ratios by which the Theorems are deduced are the same, the Conclusions must certainly be so likewise.

205. THEOREM 17. *If three Lines are Proportionals, the Rectangle under the Extremes is equal to*
I 2 *the*

* Analytically thus. The Product of the Extremes $\overline{a+b} \times \overline{c-d} = ac + bc - ad - bd$; and the Product of the Means $\overline{a-b} \times \overline{c+d} = ac - bc + ad - bd$. But it is known that $ad = bc$ by the Supposition and Art. 193; $\therefore +bc - ad$ in one Equation, and $-bc + ad$ in the other, vanish, being equal to Nothing; whence the Equations become $\overline{a+b} \times \overline{c-d} = ac - bd$, and $\overline{a-b} \times \overline{c+d} = ac - bd$ also; consequently $\overline{a+b} \times \overline{c-d} = \overline{a-b} \times \overline{c+d}$, that is, the Product of the Extremes = Product of the Means, and $\therefore$ the Quantities proportional, Q. E. D.

the Square on the Mean; that is, if $a : b :: b : c$; then $a \times c = b^2$.

- * 193. For, the Rectangle of the Extremes being $a =$ the Rectangle of the Means, $a \times c = b \times b$; but the Rectangle $b \times b$ is a Square, $\therefore a \times c = b^2$.
Q. E. D.

Note. This follows so easily from Art. 193, that it might have been given as a Corollary thereto, but was not at that Time thought of.

206. THEOREM 18. *The Rectangles under proportional Lines are Proportionals. Thus, if $a : b :: c : d$, and $e : f :: g : h$, then we affirm that $a \times e : b \times f :: c \times g : d \times h$.**

- * 153. For, $a \times e$, and $b \times e$, being a Equimultiples of
* 187. a and b , we have $a \times e : b \times e :: a : b$; but $a : b :: c : d$ by the Supposition, $\therefore$ by Equality of Ratios, $a \times e : b \times e :: c : d$. Again, $c \times g$, $d \times g$, are Equimultiples of c , d , $\therefore c \times g : d \times g :: c : d$; $\therefore$ by Equality of Ratios, $a \times e : b \times e :: c \times g : d \times g$. Also $b \times e$, $b \times f$, are Equimultiples of e , f , $\therefore b \times e : b \times f :: e : f$; but $e : f :: g : h$, by the Supposition, $\therefore$ by Equality of Ratios, $b \times e : b \times f :: g : h$.

Once again; $d \times g$, $d \times b$, are Equimultiples of g and b , $\therefore d \times g : d \times b :: g : b$. $\therefore b \times e : b \times f :: d \times g : d \times b$, by Equality of Ratios. By Alternation the Analogy (above found) $a \times e : b \times e :: c \times g$

- * This may be thus demonstrated by *Algebra*, viz. By the Supposition, as $a : b :: c : d$; and $e : f :: g : h$; hence the Product of the Extremes being $a =$ the Product of the Means, we have $ad = bc$, and $eh = fg$; $\therefore$ by multiplying each Side of the first Equation by the correspondent Term of the other Equation, we get $aedh = bfcg$, (because equal Things multiplied by equal Things must produce equal Products,) or, which is the same, $ae \times dh = bf \times cg$; $\therefore$ (by Art. 193.) as $ae : bf :: cg : dh$.
Q. E. D.
- * 193.

$c \times g : d \times g$, becomes $a \times e : c \times g :: b \times e : d \times g$; and the Analogy just found, taken alternately, is $b \times e : d \times g :: b \times f : d \times b$; $\therefore$ by Equality of Ratios, $a \times e : c \times g :: b \times f : d \times b$. Q. E. D.

207. THEOREM 19. *The Squares upon four proportional Lines are proportional; viz. if $a : b :: c : d$; then we affirm that $a^2 : b^2 :: c^2 : d^2$.*

For, if $a : b :: c : d$, and $a : b :: c : d$, then by the last Theorem, $a \times a : b \times b :: c \times c : d \times d$. Q. E. D.

208. THEOREM 20. *If four Squares are Proportionals, their Sides will be so too; that is, if $a^2 : b^2 :: c^2 : d^2$, then we affirm that $a : b :: c : d$.*

For let $a : b :: c : e$, then per last Theorem, $a^2 : b^2 :: c^2 : e^2$, $\therefore$ by Equality^a of Ratios, $c^2 : d^2 :: c^2 : e^2$, $\therefore d^2 = e^2$. $\therefore d = e$, $\therefore$ by writing d for its^a Equal e , we have $a : b :: c : d$. Q. E. D.

††† We may perhaps treat more fully of the Doctrine of Proportionals in a future *Essay*, in a new and algebraic Manner; our Intention at this Time being only to give so much as is in some Degree necessary for the present Occasion: In doing which we have been obliged to deviate greatly from *Euclid's* Method; and indeed, after all that has or can be done, to restore the genuine Demonstrations of *Euclid*, that Method will be found generally too obscure for Learners at their first setting out. And we hope those who are prejudiced to Antiquity will excuse us in saying, that, as Mathematics are now improved, we cannot see any Necessity of adhering to *Euclid's* Demonstrations of the Doctrine of Proportionals. And to speak plainly, we think many of the Propositions are in a Manner useless to an Algebraist, as many Theorems which are with Difficulty demonstrated.

demonstrated by them, may be easily investigated by the analytic Art.

Notwithstanding what Truth has obliged us to remark, we may venture to affirm, that we hold *Euclid* in the highest Veneration, and, if we are not mistaken, more justly than those who are so much bigotted to him, as to condemn any Author, who shall dare to deviate from him in the least Article. The Censures of such will therefore give us no Concern.

209. THEOREM 21. *If in a right-angled Triangle*
 Pl. IV. *ABC, right-angled at A, a Perpendicular AD be let*
 F. 13. *fall from the right Angle, to the opposite Side BC ;*
the Triangles on each Side of it shall be similar to the
whole Triangle, and to one another.

For the $\triangle ADC$ being right-angled at D, the $\angle$
 * 83. CAD is $=$ a Rt. $\angle - \angle C$, and the $\triangle ABC$ being
 Rt. $\angle$ d at A, the $\angle B =$ a Rt. $\angle - \angle C$; $\therefore \angle$
 * 45. CAD $= \angle B$; and $\angle C$ being common to both
 $\triangle$ s, and $\angle$ s ADC, BAC, Rt. $\angle$ s, the $\triangle$ s ABC,
 * 194. ADC, are equiangular, and $\therefore$ * similar. In the
 same Manner it may be shewn, that the $\triangle$ s BAD,
 ABC, are similar; and the $\triangle$ s ADC, ABD, being
 each similar to the $\triangle ABC$, they must consequently
 be similar to each other. Q. E. D.

210. COROLLARY 1. *Hence these Analogies.*

1. $AB : BC :: AD : AC.$
2. $BD : DA :: DA : DC. *$
3. $BC : AB :: A : BD.$
4. $BC : AC :: AC : DC.$

211. COROLLARY 2. *If from any Point A, in*
the Circumference of a Circle, the Perpendicular AD is
let

* This is all we mean when we say AD is a geometric mean Proportional between BD and DC.

let fall on the Diameter BC, and the Chords AB, AC, drawn from the Point A to the Extremities of the Diameter BC; as the Angle in a Semicircle is a * right $\angle$, ^{* 148.} it plainly follows, that the Rectangle of the two Chords is equal to the Rectangle under the whole Diameter and that Perpendicular; also that the Rectangle under the two Segments of the Base is equal to the Square of the Perpendicular; and lastly, that the Square of either Chord is equal to the Rectangle under the whole Diameter and adjacent Segment.*

For in four proportional Quantities, the Rectangle of the Extremes being ^a = the Rectangle of the Means, from the Analogies in Art. 210 we have, ^{* 193.}

$$\left. \begin{array}{l} 1. AB \times AC = BC \times AD. \\ 2. BD \times DC = DA^2. \\ 3. BC \times BD = AB^2. \\ 4. BC \times DE = AC^2. \end{array} \right\} \text{Q. E. D.}$$

212. THEOREM 22. If two Lines cut one another within a Circle, (as in Pl. IV. Fig. 14.) the Rectangle of the Segments of one of them is equal to the Rectangle of the Segments of the other, viz. $AE \times EC = BE \times ED$. And if two Lines cutting a Circle, being produced, intersect each other without the Circle, (as in Pl. IV. Fig. 15.) then also will $AE \times EC = BE \times ED$.

For let BC, AD, be joined; then the $\angle$ s C, and D, standing on the same Arc AB are ^a equal to ^{* 146.} each

* From hence may be deduced a very easy Demonstration of the celebrated Proposition generally cited by 47 Euclid's 1st Book, viz. our 115th Article. For, by this, $BD \times BC = AB^2$, and $DC \times BC = AC^2$; $\therefore BD \times BC + DC \times BC$ (by the 46 Art.) $= AB^2 + AC^2$; or, which is the same, $BD + DC \times BC = AB^2 + AC^2$; that is, $(BC \times BC, \text{ or } BC^2 = AB^2 + AC^2$.
Q. E. D.

- ^b 69. each other; and the $\angle BEC^b = \angle AED$, the Δ s
^c 80. BEC , AED , have two $\angle$ s of one respectively =
 two $\angle$ s of the other, and $\therefore$ the third $\angle A^c =$ the
 third $\angle B$; consequently, the Δ s are equiangular,
^d 194. and $\therefore$ similar. Hence, $AE : ED :: BE : EC$; $\therefore$
^e 193. $AE \times EC^e = BE \times ED$.

Pl. IV. 213. THEOREM 23. *If from any Point C, with-*
 Fig. 16. *out a Circle, two Lines CA, CB, are drawn in such*
Manner, that CB touches the Circle, and CA cuts it;
then will $CA \times CD = BC^2$.

- ^a 151. For let BD , BA be joined; then the $\angle DBC^a =$
 A , and the $\angle C$ common to both Δ s, $\therefore$ the Δ s
 DBC , CBA , have two $\angle$ s in one = two $\angle$ s in the
 other, $\therefore$ the remaining $\angle BDC$ in one = the re-
 maining $\angle CBA$ in the other, $\therefore$ the Δ s are equi-
^b 194. angular, and consequently ^b similar, $\therefore AC : CB ::$
^c 193. $CB : CD$; $\therefore AC \times CD^c = CB \times CB$. Q. E. D.

Pl. IV. 214. THEOREM 24. *In equal Circles, Angles,*
 F. 17. *whether at the Centers or Circumferences, have the*
same Ratio with the Circumferences on which they stand.
Thus, let G and H be the Centers of the two Circles;
then we affirm, that, as the Circumference BC : the*
Circumference EF, :: the $\angle BGC : \angle EHF$ and ::
 $\angle BAC : \angle EDF$.

For if we take any Number of Circumferences,
 CK , KL , each = BC ; and any Number what-
 ever FM , MN , each = EF ; and join GK , GL ,
 HM , HN ; then, because the Circumferences BC ,
 CK , KL , are all equal, the $\angle$ s BGC , CGK , KGL ,
 will be = each other, (for 'tis manifest they would
 exactly cover each other;) $\therefore$ whatever Multiple the
 Circumference BL is of the Circumference BC , the
 same Multiple is the $\angle BGL$ of the $\angle BGC$. And
 by the same Method of Reasoning the Circumference
 EN is the same Multiple of the Circumference EF ,
 as the $\angle EHN$ is of the $\angle EHF$. And if the Circum-
 ference

ference BL be $\equiv$ the Circumference EN, the $\angle$ BGL will be $\equiv \angle$ EHN, (as they would exactly cover each other;) and if the Circumference BL be Γ EN, the $\angle$ BGL will be $\Gamma \angle$ EHN, and if BL be Λ EN, the $\angle$ BGL $\Lambda \angle$ EHN. Here, then, are four Magnitudes, viz. the two Circumferences BC, EF, and the two $\angle$ s BGC, EHF; and of the Circumference BC, and of the $\angle$ BGC, have been taken any Multiples whatever, viz. the Circumference BL, and $\angle$ BGL; and of the Circumference EF, and $\angle$ EHF, any Equimultiples whatever, viz. the Circumference EN, and $\angle$ EHN; and it has been proved, that if the Circumference BL be either Γ , $\equiv$, or Λ the Circumference EN, the $\angle$ BGL will be respectively Γ , $\equiv$, or Λ the $\angle$ EHN; $\therefore$ the Circumference BC : Circumference EF $\ast :: \angle$ BGC : $\angle$ EHF. Q. E. D. * 156.

As to the $\angle$ s BAC, EDF, at the Circumferences, they are the $\frac{1}{2}$ Halves of the $\angle$ s at the Centers BGC, EHF; $\therefore$ as the Wholes have been shewn above to be in Proportion to the Circumferences BC, EF, their Halves must be so too. Q. E. D. b 145.

215. COROLLARY 1. *The $\angle$ BGC in the Center : 4 Rt. $\angle$ s $::$ the Circumference or Arc BC on which it stands : the whole Circumference of the Circle.*

216. COROLLARY 2. *Hence the Circumferences DE, BC, of unequal Circles, which subtend equal Angles at the Centers, as $\angle$ s DAE, BAC, are similar; viz. as Arc DE : the whole Circumference of the Circle DEF $::$ BC : the Circumference of the Circle BCG. And if they are similar, they will subtend equal Angles.* Pl. IV. F. 18.

For 1st. $\angle$ DAE : 4 Rt. $\angle$ s $\ast ::$ DE : the Circumference of the whole Circle DEF; and ($\angle$ BAC, or, which is the same,) $\angle$ DAE : 4 Rt. $\angle$ s $::$ Arch BC : the Circumference of the whole Circle BCG. * 215.

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BCG. $\therefore$ by Equality of Ratios, Arc DE : Circumference of the Circle DEF :: Arc BC : Circumference of the Circle BCG. Q. E. D.

- 215. Secondly. As $\angle DAE : 4 \text{ Rt. } \angle s^a ::$ the Arc DE : the Circumference DEF ; and $\angle BAC : 4 \text{ Rt. } \angle s ::$ Arc BC : the Circumference of Circle BCG ; but, by the Supposition, the Arc DE : the whole Circumference of the Circle DEF :: BC : the whole Circumference of the Circle BCG ; $\therefore$ by Equality of Ratios, $\angle DAE : 4 \text{ Rt. } \angle s :: \angle BAC$
 • 181. $: 4 \text{ Rt. } \angle s ; \therefore \angle DAE^b = \angle BAC$. Q. E. D.

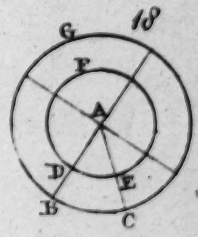
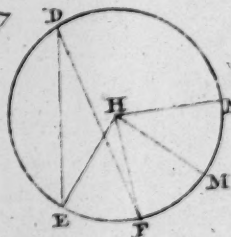
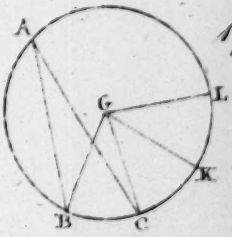
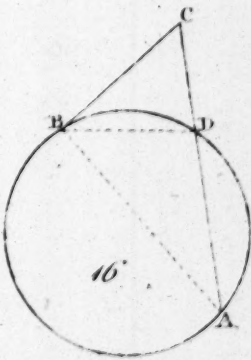
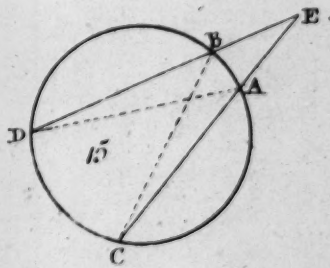
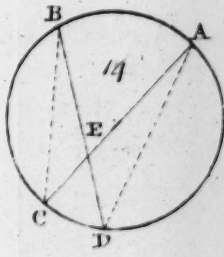
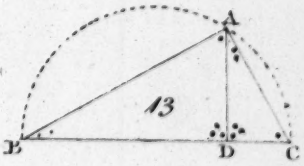
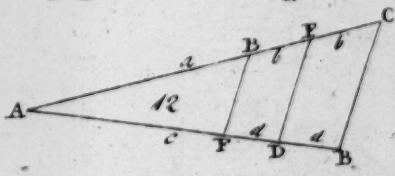
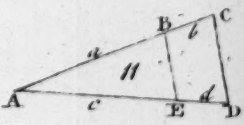
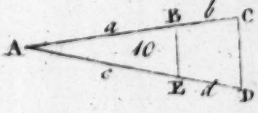
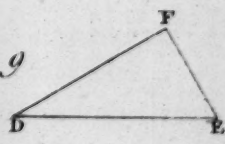
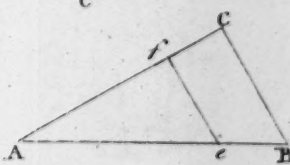
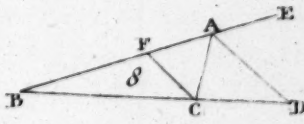
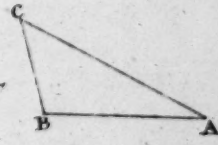
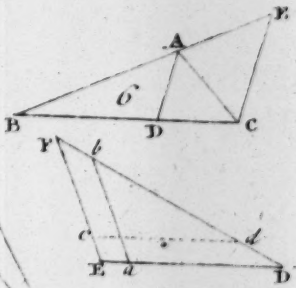
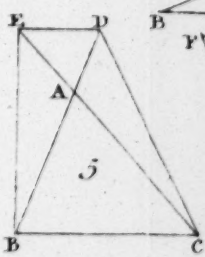
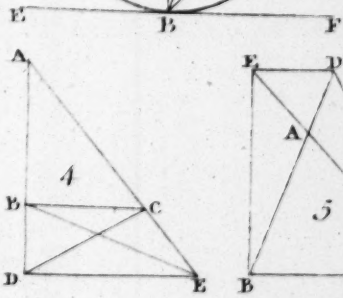
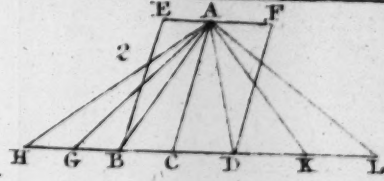
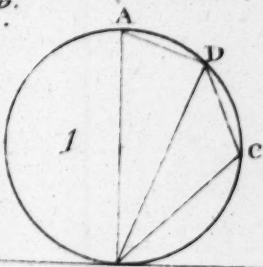
217. *Scholium.* Mathematicians, for the more easy comparing the Proportion which Angles bear to each other, suppose the Circumference or Periphery of every Circle to be divided into 360 equal Parts, which they call Degrees, (so that 180 Degrees are a Semicircle, or 2 Rt. $\angle s$, and 90 Degrees a Quadrant, or Rt $\angle$;) each Degree is supposed to be divided into 60 equal Parts, called Minutes, and each Minute into 60 equal Parts, called Seconds, &c. — Hence it is plain, that in unequal Circles, as well as in equal, equal Angles are subtended by an equal Number of Degrees. For, as a Degree is $\frac{1}{360}$ Part of the Circumference of any Circle, the Length of a Degree will increase, or decrease, in the same Proportion as the Circumferences of the Circles ; and so the Arch, representing any Number of Degrees in one Circle, will be to the whole Circumference, as the Arch representing the same Number of Degrees in any other Circle, to the whole Circumference of that other Circle, and $\therefore$ will, by the last Corollary, subtend equal $\angle s$. Hence appears the Reason of the Method of denoting an $\angle$ by saying it contains so many Degrees.

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218. **I**T will now be convenient for the Learner to procure a Case of Pocket Instruments, for his more easy performing the following Problems.

The Case we would recommend ought at least to contain 2 Pairs of Compasses, a Drawing-Pen and Point, a Pencil and Point, Parallel Ruler, Ivory Plotting-Scale, Ivory Sector, and Brass Protractor. To these may also be added, a Pricking or Dotting-Wheel-Point, a Protracting-Pin, a Pair of Bows, Tracing-Point, Feeder, &c.

219. *Compasses* are so well known to every one, that it would be lost Labour to describe them; however, it may be of Use to some Persons to hint, that one of the Compasses hath one Leg to take out, that in its Stead may be occasionally used, the Drawing-Pen-Point, a Pencil-Point, and Dotting-Wheel-Point. — In choosing Compasses take Notice, that in opening and shutting, the Legs move not by Starts, but that the Motion in the Joint be * easy and regular; and that the Points are of an equal Length, &c. For lessening the irregular Friction, (if any,) the Head of the Compasses may be warmed, and a little || yellow Wax melted and poured between the Joints.

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* And for this End let Part of the Joint be Steel, for Brass and Steel will wear more equable than either singly. Also let the Pin or Axle of the Joint be a Steel Screw, by Means of which you can make your Compasses to move in the Joint stiffer or easier at Pleasure.

|| Or, if the Wax is too stiff of itself, a little Wax and Oil mixed together.

The Solder for cementing the Steel Points to the Brass, is generally made with twice as much Silver as Copper ; which must be melted in a Crucible, and, when cold, hammered to about the Thickness of a Card. Solder is also frequently made by melting 3 Parts of Copper with one of Zink.

In Order to make the Solder run the better between the Parts to be soldered, Borax, bruised very fine, is laid on the Parts to be cemented, and melted together with the Solder.

To one Point of the smaller Compasses is sometimes fixed in the Shank, a Spring, which, by Help of a Screw, when you have opened the Compass nearly to the required Distance, moves it very regularly and gradually, without any starting to the Point intended, for the more accurately taking off any required Distance. Thus, holding the Compasses in the left Hand, (with the Screw turned towards the right,) if with the other you turn the Screw backward, the Spring Point will approach nearer to the other Point ; but if you turn the Screw forward, the Spring Point will be farther distant from the other : However, after the Learner has been some Time used to a Pair of Compasses, he will not stand in much Need of this Contrivance.

The lesser Compasses are used in transferring the Measures of Distances from one Place to another, or in describing Circles, or Parts of Circles. —

The larger Compasses are chiefly intended for describing Circles, or Parts of Circles, either obscurely by the Steel Point, in Ink by the Drawing-Pen-Point, in Lead by the Pencil Point, or dotted by the Dotting-Wheel-Point ; for either of these, as before observed, may be fixed in the Shank of the Compasses. The *Dotting-Wheel-Point* is not absolutely necessary, for dotted Lines are better made by the Drawing Pen or Point, (though something slower,) for it is but seldom the Dotting-Wheel-Point is so well made as to perform accurately.

As almost every Learner, at his first Beginning, handles a Pair of Compasses very awkwardly, it may be proper to caution him by all means to endeavour to make Use of them with the right Hand only, never using the left, unless in a Case of Necessity, which after some Practice he will seldom have Occasion for.—In using the Compasses be always careful to keep them nearly upright.

The *Bow* is a small Kind of Compasses, one of whose Legs is as common, the other is a Drawing Pen Point : Its Use is to describe Arcs or Circles of short Radii, which could not be so readily and accurately done by the large Compasses and Drawing-Pen-Point.

The *Drawing Pen Point* and *Pencil Point* (or Crayon Point, as it is frequently called,) in the best Cases have generally a Socket fitted to them, that they may fill up but one of the Holes or Partitions of the Case. These Points, and also the Dotting-Wheel-Point, have a Joint in them just under that Part which locks into the Shank of the Compasses, that the Part below the Joint may stand perpendicular, or nearly so, to the Paper, or Plane, on which the Arc is to be described ; otherwise it will not be smooth.

Of the larger Compasses, those are esteemed the best, whose moveable Points are locked in by a Spring and Catch fixed in the Shank ; for if this Spring be well effected, the Point is thereby kept firm and steady ; the contrary of which frequently happens when the Point is kept in by a Screw in the Shank.

If, instead of the larger Compasses being made with shifting Points, there were two Pairs put into the Case, to one of which the Ink Point was fixed, and to the other the Pencil Point, this, as Mr. Robertson justly * observes, would save the Trouble
of

* In his Treatise of Math. Instruments, &c.

of changing the Points in the Compass at every Time they were used, and the additional Expence, or Bulk of the Case, would be very little.

The *Black-lead Pencil* is of great Use to describe the first Sketch of a drawing before it is marked with Ink; because any false or superfluous Lines may be rubbed out with a Piece of stale Bread.

Black-Lead is produced in many Countries; but the best yet discovered is found in the North of *England*; it is dug out of the Ground in Lumps, and sawed into Scantlings proper for Use. If it will not bear paring to a Point with a sharp Pen-knife, it is not fit for drawing on Paper. There are three Sorts of *Black-lead*; the softest is proper for taking rough Sketches, the middling for drawing of Landscapes and Ornaments, and the hard for drawing mathematical Figures.

The *Feeder* is a thin flat Piece of Brass, or Silver, commonly fixed to a Cap that is put on the Pencil: Its Use is either to put Ink between the Blades of the Drawing Pen, or to pass it between the Points when the Ink by drying does not flow freely.

The *Drawing-pen* is used only for drawing right Lines: It consists of two Blades with Steel Points, fixed to a Handle; the Blades being a little bent, by Means of a Screw, cause the Steel Points to approach as near together as you please, so that a Line may be drawn stronger or finer at Pleasure. When you use this Instrument put the Ink between the Blades, either with a common Pen or the Feeder, and by moving the Screw bring the Points to a proper Distance; then hold the Pen a little inclined, but take Care that both Points touch the Paper, by which Means you may draw a neat Line equally broad in all Parts; which is not easily done by a common Pen. — Before you put the Pen into the Case after using, it will be necessary to clean the Points from the Ink, otherwise they will be inclined to rust, &c. If one of the Blades is made to move on

on a Joint, as the *Bow* generally is, they would be more easily cleaned.

The *Protracting-pin* is a small Piece of sharp Steel much resembling the Point of a Needle, fixed into the lower End of the Top Part of the Handle of the *Drawing-pin*; it screws for Preservation into the lower Part of the Handle: Its Use is to set off Angles from the Protractor, (as will be shewn hereafter,) and to mark off the Intersection of Lines, that the Places of crossing may be the better seen.

220. We think it needless to describe the *Tracing Point* in this Place, as it will come in more properly in another; and the Construction of the other Instruments will be shewn, as they occasionally become necessary or useful. We shall therefore, in this Place, only observe, that as a Ruler must be very strait to draw right Lines with, it may be examined by drawing a right Line with it, and then turning the Ruler about End for End, apply the same Side to the Line, and, if the Ruler exactly coincides with the Line, the Ruler is strait, but if not, you will see where the Fault is. A good Eye will also discover whether a Ruler is strait by only looking over the Edge of it.

221. PROBLEM 1. *To draw a Line from a given Point C, equal to a given Line AB.* P.V.F.1.

Take a Pair of Compasses, and placing one Point in A, open the other Point to B; then will the Distance between the Points of the Compasses Legs be $=$ the given Line AB; $\therefore$ setting one Leg in the Point C, with the other make another Point D, and by placing the Edge of a Ruler to touch the Points C, D, draw the right Line CD, and it is manifest the Thing is done; viz. $CD = AB$.

222. PROBLEM 2. *From a greater Line, AB, to cut off a Part equal to a lesser Line C.* Fig. 2.

This

This may be easily done by opening the Compasses equal to the Length of the Line C, (as shewn in the 1st Problem,) and laying that Distance from A to D on the Line AB, so is $AD =$ the Line C. Which was to be done.

Fig. 3. 223. PROBLEM 3. *About a given Point, A, to describe a Circle whose Radius shall be equal to a given Line B.*

Open the Compasses $=$ the Line B; then resting the Point of one Leg in A, carry the other about from where it falls, as C, till it arrives again at that Point, and it is manifest the required Circle will be described by the moving Point.

N. B. In describing of Circles be careful not to press the Paper, either at the Center or under the moveable Point, with more than just the Weight of the Compasses, to avoid making great Holes or Blots, which frequently happen to young Persons at their first beginning to practice.

Fig. 4. 224. PROBLEM 4. *To make a Triangle whose Sides shall be respectively equal to three given Lines.*

Let the three given Lines be A, B, C; take DE $=$ C, then about D, as a Center, with a Radius $=$ B; describe Part of a Circle, viz. Arc ab, and take DE $=$ C, and about E, with a Radius $=$ A, describe Part of the Circumference of a Circle, viz. the Arc dc; then from the Point of their Intersection draw the Lines FD, FE, and the $\triangle DFE$ will be that required; for by the Construction, $DE = C$, $DF = B$, $EF = A$.

Note. $A + B$ must be $> C$, otherwise the Arcs will not intersect, and so the Problem would be impossible.

225. PROBLEM 5. *To bisect a given Angle BAC, Pl. V. that is, to divide it into two equal Angles.* F. 5.

CONSTRUCTION. With any Radius AD, sweep the Arc DE; then with any Radius $r \frac{1}{2}$ the Distance of the two Points D, E, about the Points D and E, as Centers, sweep the Arcs *cd*, *ab*, and from the Point of Intersection F, draw the Line AF, and it is done; viz. the $\angle FAC = \angle FAB$.

DEMONSTRATION. AD being = AE, and DF = EF, by the Construction, and AF common to both Δ s DAF, EAF, the Δ s are themselves ^a equal ^a 95. in all Respects; $\therefore \angle DAF = \angle EAF$. Q. E. D.

226. PROBLEM 6. *To bisect a given finite Line Pl. V. AB, that is, to divide it into two equal Parts.* F. 6.

CONSTRUCTION. About the Points A and B, as Centers, with any Radius $r \frac{1}{2}$ AB, describe Parts of Circles, viz. the Arcs *ab*, *cd*, and join the Points of Intersection E, F; and the Line EF will cut the Line AB in G, the Point which was required.

DEMONSTRATION. For by the Construction, AE = AF = BE = BF, and EF common; $\therefore$ the Δ s EAF, EBF, have 3 Sides of one respectively = 3 Sides of the other, and $\therefore$ the Δ s are themselves ^a equal in all Respects, and $\therefore \angle AEG = \angle$ ^a 95. BEG. Hence the Δ s AEG, BEG, have two Sides AE, EG, of one = two Sides BE, EG, of the other, and the contained $\angle$ AEG of one = the contained $\angle$ BEG of the other, $\therefore$ the Base AG ^b = ^b 58. the Base BG. Q. E. D.

227. COROLLARY. *This Problem serves also to raise a Perpendicular on the Middle of a Line.*

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For

For we have just shewn that the Point G is in the Middle of the Line, viz. $AG = BG$; also $AE = BE$, and EG common; $\therefore$ the Δ s AEG , EBG , are $\therefore$ equal in all Respects, and $\therefore \angle AGE = \angle BGE$; $\therefore EG \perp$ to AB . Q. E. D.

Pl. V. 228. PROBLEM 7. *To draw a Perpendicular to a given Line AB , from any Point C in that Line.*

CONSTRUCTION. From the Point C set off any Distance towards A , as CD ; and from the same Point the same Distance CE towards B . Then with any Extent of the Compasses r CD , putting one Leg in D , with the other sweep the Arc ab , and with the same Extent of the Compasses, placing one Leg in E , with the other sweep the Arc cd ; then from the Point of Intersection F , draw the Line FC , and it is the $\perp$ which was required.

DEMONSTRATION. CD being $= CE$, and $DF = EF$, by the Construction, and FC common, the Δ s DCF , ECF , have 3 Sides of one respectively $=$ the three Sides of the other; and $\therefore$ the Δ s themselves are equal in all Respects, and consequently the $\angle DCF = \angle ECF$. Q. E. D.

Pl. V. 229. PROBLEM 8. *To draw a Perpendicular to a given Line AB , of an unlimited Length, from a given Point C , without it. Or, in other Words, to let fall a Perpendicular from a Point above to a Line beneath.*

CONSTRUCTION. With any Radius of a sufficient Length to intersect the Line AB , with one Leg of the Compasses in C , with the other sweep the Arc DE ; then with any Extent of the Compasses r $\frac{1}{2} DE$, one Leg resting in D , with the other sweep the Arc ab ; in like Manner, putting one Leg in E , with the other sweep the Arc cd , then from the

the Point of Intersection, F, draw a Line to C, and it will be the $\perp$. Q. E. I.

DEMONSTRATION. For DC being = CE, and DF = EF, by the Construction, and CF common, the Δ s DCF, ECF, have 3 Sides of one respectively = 3 Sides of the other, and $\therefore$ they are = each other in all Respects ; Consequently the $\angle$ DCG = $\angle$ ECG. Now DC being = EC, GC common, and the contained $\angle$ DCG = the contained $\angle$ ECG, as hath been just shewn, the Δ s DGC, EGC, are = each other in all Respects ; and consequently the $\angle$ DGC = $\angle$ EGC. Q. E. D. * 95.

Otherwise thus.

CONSTRUCTION. Join A, C, and = bisect the Line AC in E ; then with the Radius EA, or EC, sweep the Semicircle CDA, and where it intersects the Line AB in D, draw the Line CD, and it will be that required. * 226.
Pl. V.
Fig. 9.

DEMONSTRATION. For the $\angle$ ADC in a Semicircle is a * Rt. $\angle$. Q. E. D. * 148.

230. PROBLEM 9. *At a given Point A, in a given Line AB, to make an Angle equal to a given Angle EDF.* Pl. V.
Fig. 10.

CONSTRUCTION. About the Points A and D, with any Radius, describe the Arcs ab , cd , then take with the Compasses the Distance between the two Points, d , c , which set off from b to a , on the Arc ab , and thro' the Point a draw the Line AC ; then is the $\angle$ BAC = $\angle$ EDF.

DEMONSTRATION. For let the Points d , c , and b , a , be joined ; then by the Construction the 3 Sides of the Δ bAa are respectively = the 3 Sides of the Δ dDc ,
L 2 dDc ,

dDc , and $\therefore$ the Δ s themselves are equal in all Respects ; consequently the $\angle A = \angle D$. Q. E. D.

231. PROBLEM 10. *To construct and shew the Use of a Protractor, that is, a Semicircle divided into Degrees, for setting off or measuring Angles.*

Pl. V. CONSTRUCTION. Let ADB be a semicircular
Fig. 11. Piece of Brass. Now to divide the Circumference ADB into 180 equal Parts, called Degrees, we proceed thus. On the Center C $\perp$ AB draw ^a CD, so will the $\angle ACD$ or BCD be a Rt. $\angle$, or the Arc AD = 90 d. = BD. Then take the Radius AC in the Compasses, and set it from A to E, and from B to F, then is AE, EF, FB each = 60 Degrees ; and setting the same Distance from D to G, and from D to H, the Arcs DG, DH, are each = 60 d. and so AG, GE, ED, DF, FH, HB, each = 30 Degrees. Hence, by dividing each of these Arcs into 3 equal Parts, (by a Trial or two, or by Art. 241.) the Circumference will be divided into every 10 Degrees ; which by taking the Halves will be divided into every 5 Degrees, and by a Trial or two the 5th Part of these last Divisions may be taken, and so the Semicircle divided into every single Degree, as represented in the Figure.

DEMONSTRATION. The only Thing requiring a Demonstration in this Construction is, that the Radius is = the Chord of 60 Degrees ; or, that the Compasses being opened = the Radius, one Leg resting in the Point A, the other will fall on the Circumference in E, the Point of 60 Degrees. For suppose A, E, joined, then by the Construction,
60. the ΔACE would be equilateral, and $\therefore$ ^a equi-
85. angular ; $\therefore$ the $\angle ACE =$ ^b $\frac{1}{3}$ of 2 Rt. $\angle$ s, that is, $\frac{1}{3}$ of 180 Degrees = 60 Degrees. Q. E. D.

232. *The 1st Use of the PROTRACTOR is, to make an Angle equal to any Number of Degrees.*

For Example, at the Point C, on the Line IC, to make an $\angle = 60$ Degrees. Lay the Diameter AB on the Line IC, so that the Center of the Protractor may be at the angular Point C; then against 60 Degrees, (*viz.* the Point E,) make a * Dot, and, taking off the Protractor, draw the Line CK, to pass thro' the Point E, and the $\angle KCI$ will be that required.

233. *Secondly. To measure an Angle.*

This is only the Reverse of the last Article, and is done by laying the Protractor as there directed, and it will be seen by Inspection what Number of Degrees is cut by the Line KC. — All this is very plain.

234. PROBLEM II. *To construct and shew the Use of a Line of Chords.* Pl. V.
Fig. 12.

Definition. A Line of Chords is only the Chords taken from a Quarter of a Circle and set off on a Rt. Line.

Hence this CONSTRUCTION. Set one Point of the Compasses in the Point A, Fig. 11, and open the other Leg to any Number of Degrees, for Example Sake, to G, 30 Degrees; then that Distance laid from *a* towards *c*, on the Line *ac*, Fig. 12. will give the Point *b* for 30 Degrees. After the same Manner

* For this there is in the Drawing Pen a Point like a Pin, called a *Protracting Pin*. Another Kind of Protractor is made by transferring the Degrees from the Arch to the Sides of a Scale, or Ruler, as is intended to be fully described in another *Essay*.

Manner the other Degrees as far as 90 are laid on the Line *ac*, which then is called a Line of Chords. This is too manifest to need a Demonstration.

235. *The Use of a Line of Chords is very easy. For 1st. To make an Angle equal to any Number of Degrees, the Chord of 60 Degrees being = the Radius, as demonstrated in the last Problem, with that Extent, (commonly called a Sweep of 60) describe an Arc of a Circle of a sufficient Length; then take off from the Scale (Fig. 12.) the Chord of the Number of the Degrees the $\angle$ is to contain, and set off that Distance on the Arc; then draw a Line from the angular Point, as in Art. 232, &c. To measure an Angle being only the Reverse of this, we shall omit particular Directions as unnecessary.*

Note. If the $\angle$ is more than 90 Degrees, first set off the Chord of 90 Degrees on the Arc, and from the Point where that falls, set off the Chord of the Excess of the $\angle$ above 90 Degrees.

PL. V. 236. PROBLEM 12. *To draw a Line through a given Point A, parallel to a given Line BC.*

CONSTRUCTION, &c. Take any Point D in the Line BC, and join the Points A, D; then make
 230. the $\angle EAD^a = \angle ADC$, and the Line EF will be
 73. $\parallel$ to BC. Q. E. I. & Q. E. D.

PL. V. 237. PROBLEM 15. *At any given Distance to F. 14. draw a Line parallel to another given Line AB.*

CONSTRUCTION. In the Line AB take any
 228. Point C, and erect $CE^a \perp$ to AB; then on it set off the given Distance from C to G, and make the
 230. $\angle KGC^b = \angle BCG$, then will the Line IK be $\parallel$
 40. to AB. Q. E. I. & Q. E. D.

238. *Scholium.* This and the last Problem may be performed something more expeditiously, though not strictly geometrical, by the following Methods.

PROBLEM 12th *may be thus constructed.*

Pl. V.
Fig. 15.

Setting one Leg of the Compasses in the Point A, open the other so as, being turned about, it may touch, but not cut the Line BD (as represented by the Arc *ab*;) then take any Point C in the Line BD, and placing one Point of the Compasses in it, with the same Extent of the Compasses, with the other Leg describe the Arc *cd*; then lay a Ruler so as to touch the Arc *cd* and the Point A, and draw the Line EA; which will be that required.

The 13th PROBLEM *may be thus constructed.*

With a Radius = the given Distance, one Leg of the Compasses being in C, (a Point taken any where in the Line AB,) with the other describe the Arc *ab*: Again, take another Point D in the Line AB, and putting one Leg of the Compasses in it, with the other Leg and the same Radius describe the Arc *cd*; then applying a Ruler to touch these two Arcs, draw the Line EF for that required. These two Methods of drawing parallel Lines are grounded on the * Theorem that parallel Lines are every where equidistant.

Pl. V.
Fig. 16.

* 96.

239. PROBLEM 14. *To construct a parallel Ruler for the more ready drawing parallel Lines.*

CONSTRUCTION. Let AB, CD, be two Rulers, very straight, and every where of an equal Breadth, joined together by two Brads or Silver Pieces, EG, FH, of equal Length, moveable about the Pins E, F, G, H. Note, the Distance between the Pins EF

Pl. V.
Fig. 17.

EF must be = the Distance between the Points, G, H; then will the Figure represent a parallel Ruler.

DEMONSTRATION. For whatever Distance we open the Rulers AB, CD, to, they will always be || to each other. For, EG being = FG, and EF = GH, by the Construction, and EH common, the Δ s EGH, EFH, have 3 Sides of one respectively = the 3 Sides of the other, and $\therefore$ the Δ s are equal in ^a all Respects; consequently the $\angle$ GHE = $\angle$ FEH, and $\therefore$ AB ^b || to CD. Q. E. D.

240. Figure 18th represents another Method of making a parallel Ruler. The Brass Pieces EH, GF, are of equal Length, joined together exactly in the Middle by a Pin I, about which they are moveable, in the same Manner as a Pair of Scissars. The Points F and H are moveable about the Pins F, H; and the Ends, E and G, move in Slits made in the Ends, A and C, of the Rulers.

DEMONSTRATION. That the Rulers, AB, CD, being opened to any Distance, will always be || to each other, will appear by considering that FI, HI, EI, GI, are = each other, by the Construction, and the $\angle$ EIF ^a = $\angle$ GIH, and $\therefore$ the Δ s EIF, GIH, are ^b equal in all Respects; and consequently the $\angle$ FEI = $\angle$ IHG, and $\therefore$ the Rulers AB, CD ^c || to each other. Q. E. D.

SCHOLIUM. *The Manner of Using a parallel Ruler,* when a Line is to be drawn parallel to a given Line, at a Distance not exceeding that to which the Ruler will open, seems too evident to need any Description: But when the given Point thro' which the parallel Line is to be drawn, is at a greater Distance from the given Line than the Ruler can be opened, in such Case it may be convenient to hint, that having placed the Edge of one of the Rulers, suppose CD, on the given Line, and opened the Ruler

Ruler as much as may be towards the given Point, keep the Ruler AB, which is nearest that Point, firm in its Position, and then bring the other Leg CD close to it; where stopping it, move forward the Ruler AB, and thus proceed, alternately moving each Ruler carefully, (so as to continue always parallel to each other,) until the Ruler AB touches the given Point; when the required Line may be drawn by the Edge of that Ruler.

241. PROBLEM 15. *To triseſt an Angle ACB, Pl. V. that is, to make an Angle equal to one third of a given F. 19. Angle.*

CONSTRUCTION. About C, as a Center, with any Radius describe the Semicircle DEH, and produce BH at Pleasure. Then prick a fine Pin in the Point E, and on the Edge of a Ruler GI set off GF = the Radius of the Semicircle; then move the Ruler so that the End G may always be in the Line BH produced, and the Edge of the Ruler also touch the Pin E; and observe when the Point F on the Edge of the Ruler touches the Circumference of the Semicircle; then stop the Ruler in that Position, and draw the right Line GE; then will the $\angle FCH$ be $\frac{1}{3}$ of the $\angle ECD$, or, which is the same, the Arc HF = $\frac{1}{3}$ of the Arc DE. Q. E. I.

Note. This Problem is impossible to be constructed by a Method strictly geometrical, either by right Lines, or Circles; which is our Reason for giving the above mechanical Method.

DEMONSTRATION. Let the Number of Degrees contained by the $\angle FGC$ be denoted by a ; then, FG being = FC by the Construction, the $\angle FCG = \angle FGC = a$ Degrees; $\therefore \angle FCG + \angle FGC = a + a = 2a$. But the $\angle EFC$ being the external $\angle$ of the $\triangle GFC$ is $^b =$ the two internal and opposite $\angle$ s, $\therefore = 2a$; but FC = EC by the Construction, $\therefore$ the $\triangle FCE$ is Isosceles, and $\therefore \angle EFC ^b = \angle FEC$

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 $= 2a$ * 59^db 76^d

^c 78. $= 2a$ by the above. Now the $\angle FCE^c = *180^\circ$ minus the Sum of the two $\angle$ s EFC, and FEC, that is, $= 180^\circ - 4a$. But the Arcs HF, FE, are the Measures of the $\angle$ s GCF, and FCE, respectively; $\therefore$ the Arc HE $=$ FE + Arc HF $= 180^\circ - 4a + a$, that is, $= 180^\circ - 3a$ (for one a being to be added, and $4a$ subtracted, it is manifestly the same if we only subtract $3a$.) Now a Semicircle being $= 180^\circ$ the Arc ED must be $= 180^\circ$ minus Arc HE, that is, $= 180^\circ$ minus $180^\circ - 3a$, that is, $= 180^\circ - 180^\circ + 3a$, (to subtract a Negative being the same as to add an Affirmative) $= 3a$; Hence ED $= 3$ Times HF, or, which is the same, the $\angle FCG = \frac{1}{3}$ of $\angle ACB$. Q. E. D.

Pl. V. 242. PROBLEM 16. *To describe a Parallelogram*
Fig. 20. *that shall be equal to a given Triangle ABC, and have one of its Angles equal to a given Angle D.*

^a 226. CONSTRUCTION. Bisect ^a BC in E, (and join AE) and at the Point E, in the Line EC, make the
^b 230. $\angle CEF^b = \angle D$; then through the Point A draw
^c 236. AG ^c $\parallel$ to EC, and also CG $\parallel$ to EF; then is EFGC the Parallelogram which was required.

DEMONSTRATION. The $\angle ECF$ being $= \angle D$, CG $\parallel$ to EF, and FG $\parallel$ to EC, by the Construction, the Figure EFGC is a ^d Parallelogram, and one
^d 97. of its $\angle$ s CEF $=$ the given $\angle D$. And because BE $=$ EC, the $\triangle AEB^e = \triangle AEC$, and the $\triangle AEC$
^e 109. and Parallelogram EG, standing on the same Base and between the same Parallels, the Parallelogram
^f 106. is ^f double of the $\triangle AEC$, and consequently $=$ the whole $\triangle ABC$. Q. E. D.

Pl. V. 243. PROBLEM 17. *To apply a Parallelogram to*
Fig. 21. *a given Line AB, which shall be equal to a given Triangle C, and have one of its Angles equal to a given Angle D.*

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* By 180° is signified 180 Degrees.

CONSTRUCTION. Make the Parallelogram BEFG = the ΔC , and having the $\Delta EBG^a = \angle D$, so that BE be in the same Line with AB. Then produce FG to H, and through A draw $AH^b \parallel$ to GB or FE, and join the Points H, B, and produce FE, HB, till they meet each other, and let K be the Point where they meet: Draw KL $\parallel$ to EA or FH, and produce GB, HA, to M and L; then will ABML be the Parallelogram which was required. a 242.
b 236.

DEMONSTRATION. For KL being $\parallel$ to FH, and FK $\parallel$ HL, by the Construction, the Figure FKLH is a Parallelogram, whose Diameter is HK, and AG, ME, are Parallelograms about HK, and LB, BF are the Complements; $\therefore BL^d = BF$; but $BF = \Delta C$ by the Construction, $\therefore BL^e = \Delta C$, and $\angle EBG = D$ by the Construction, and $\angle ABM^f = \angle EBG$, $\therefore \angle EBG^g = \angle ABM$; consequently the Parallelogram BL is = the ΔC , and has one $\angle ABM =$ the $\angle D$. Q. E. D. c 97.
d 113.
e 45.
f 69.

244. PROBLEM 18: To describe a Parallelogram equal to a given rectilineal Figure ABCD, and having an Angle equal to a given rectilineal Angle E. Pl. VI.
Fig. 1.

CONSTRUCTION. Join BD, and make the Parallelogram FH^a = the ΔADB , and having the $\angle HFK =$ the $\angle E$; and to the Line GH apply the Parallelogram GM^b = the ΔBDC , having the $\angle GHM = \angle E$; then will the Figure FLMK be the Parallelogram required. a 242.
b 243.

DEMONSTRATION. For the $\angle$ s FKH, GHM, being each = $\angle E$, by the Construction, it is manifest KM is one Rt. Line, and $\therefore$ the Parallels to it, FG, GL, passing through the same Point G, will be one Rt. Line FL; and GM being a Parallelogram, LM is $\parallel$ to GH, and, for the same Reason, FK $\parallel$

to GH, and consequently LM, FK are Parallels, and $\therefore$ the Figure FLMK a Parallelogram, having the $\angle FKM = \angle E$; and the Parallelogram FH being $= \triangle ADB$, and Parallelogram GM $= \triangle BDC$, the whole Parallelogram FLMK must be $=$ the whole Figure ADCB. Q. E. D.

245. SCHOLIUM 1. After the same Manner it is manifest a Parallelogram may be described, equal to any given rectilineal Figure, how many soever the Sides are; for Example, if the given Figure contained one $\triangle$ more, it would be only to apply a Parallelogram to the Line LM equal to that third $\triangle$, and having one $\angle = \angle E$, by Prob. 17.

Pl. VI.
Fig. 2.

246. SCHOLIUM 2. By this Problem may be easily found a Parallelogram equal to the Excess of one right-lined Figure given, above another right-lined Figure given. Thus, if to any Line, CD, a Parallelogram DF is applied $= A$, and to the same Line CD a Parallelogram DH $= B$, it is manifest their Difference, or the Parallelogram GF will be $=$ the Difference between the Figures A and B.

SCHOLIUM 3. Hence also it is manifest how to find a Parallelogram $=$ the Sum of any two rectilineal Figures. For if the Parallelogram DH be made $= A$, and on GH the Parallelogram GF $= B$, and have an $\angle HGE = \angle CDG$, the Parallelogram DF must be $=$ Parallelogram DH + Parallelogram GF, or, which is the same, $A + B =$ Parallelogram DF.

Pl. VI.
Fig. 3.

247. PROBLEM 19. To describe a Rectangle whose Sides shall be equal to two given Lines AB, EF.

CONSTRUCTION. On one of the given Lines AB, at one Extremity A, erect the Perpendicular AD; then take AB in the Compasses, and setting one Leg in D, with the other describe the Arc *ab*; after the same

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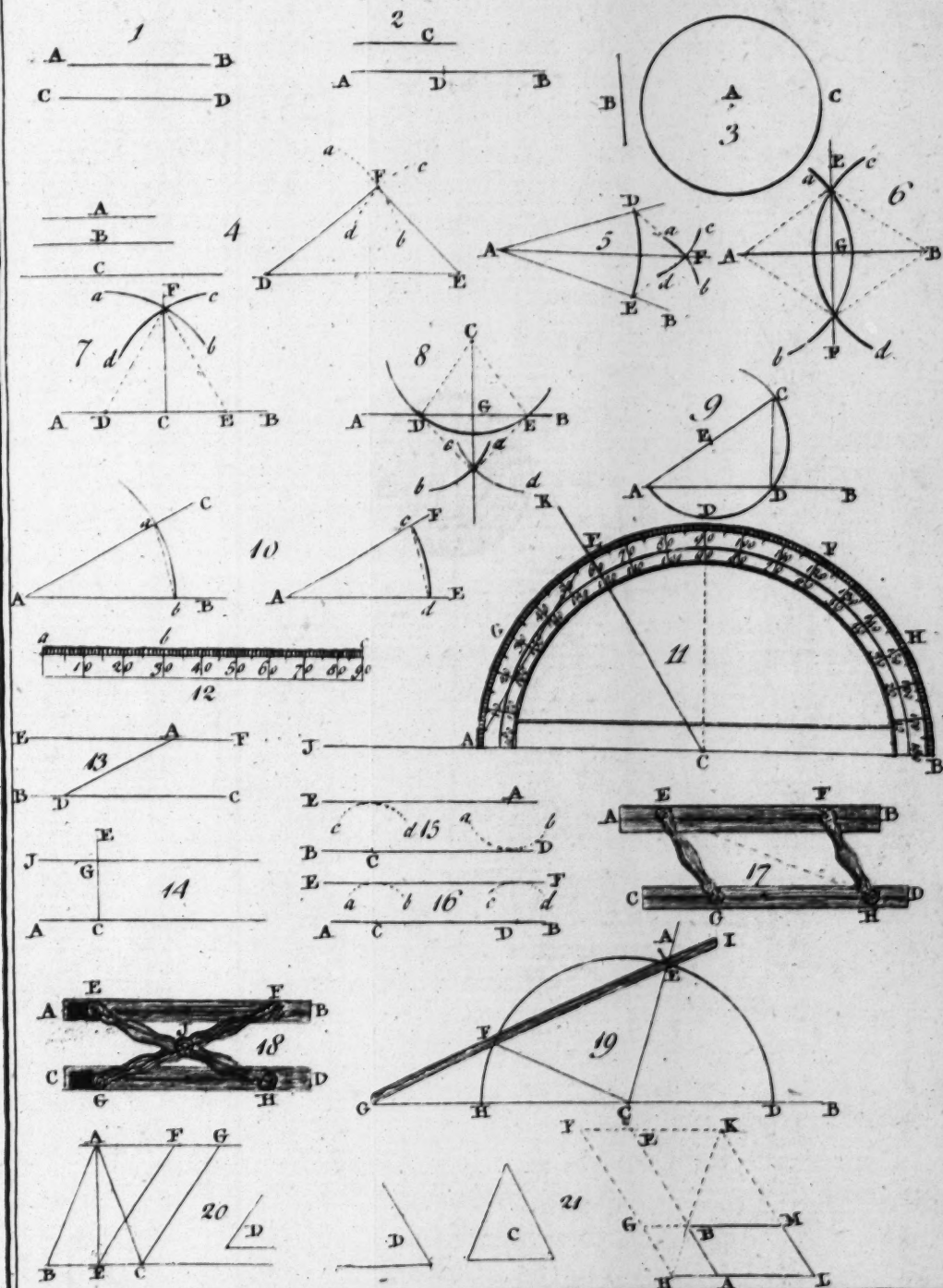
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same Manner, with the Extent of EF, and one Leg in B, with the other sweep the Arc *cd*, and let C be the Point of Intersection; then join DC, BC, and the Thing is done,

DEMONSTRATION. The opposite Sides being equal by the Construction, the opposite $\angle$ s are also equal; and $\therefore$ the $\angle$ DAB being a Rt. $\angle$ by the Construction, the other $\angle$ s must be so too: And $\therefore$ the Figure ABCD is a ^b Parallelogram, and is that required, because its Sides are by the Construction = the given Lines. Q. E. D.

248. SCHOLIUM. Hence we have an easy Method for constructing a Square; for if EF and AB are equal, the Rectangle will be a ^c Square.

249. PROBLEM 20. To make a Square equal to any Number of given Squares, e. g. equal to three given Squares, whose Sides are equal to the Lines A, B, C.

CONSTRUCTION. Draw two indefinite Lines ED, DF, at Rt. ^d $\angle$ s to each other, and take DG = A, and DH = B, and join GH, (and the Line DH will be the Side of a Square = the two Squares whose Sides are A and B.) Again, take DI = GH, DK = C, and join KI, then will KI be the Side of a Square = the three given Squares. Q. E. I.

DEMONSTRATION. For $GH^2 = GD^2 + DH^2$; but DG = A, and DH = B, by the Construction, and $\therefore DG^2 = A^2$, and $DH^2 = B^2$, consequently $GH^2 = A^2 + B^2$. Again, $KI^2 = DK^2 + DI^2$; but $DI^2 = (GH^2 =) A^2 + B^2$ by the above, and $DK^2 = C^2$, by the Construction, consequently, $KI^2 = A^2 + B^2 + C^2$. Q. E. D. — After the same Manner may a Square be constructed = any Number of given Squares.

Pl. VI. 250. PROBLEM 21. *To make a Square equal to*
Fig. 5. *the Difference of two given Squares, whose Sides are*
AB and CD.

CONSTRUCTION. On one Extremity B, of the shortest Line AB, raise a $\perp$ BE, then taking CD in the Compasses, put one Leg in A, and with the other describe the Arc *ab*, intersecting BE in F; draw AF, and it will be the Side of the required Square.

DEMONSTRATION. For $AB^2 + BF^2 = AF^2$,
 115. $\therefore$ taking AB^2 from both Sides, we have $BF^2 =$
 47. $AF^2 - AB^2$; but $AF = CD$ by the Construction,
 45. $\therefore AF^2 = CD^2$, consequently $BF^2 = CD^2 - AB^2$.
 Q. E. D.

Pl. VI. 251. PROBLEM 22. *To make a Triangle equal to*
Fig. 6. *any given quadrilateral Figure ABCD.*

CONSTRUCTION. Draw the Diagonal BD, and $\parallel$ to it draw CE intersecting AD produced in E; then join EB, and the $\triangle$ ABE will be that which was required.

DEMONSTRATION. For the $\triangle$ s DCB, DEB, being on the same Base BD, and between the same Parallels, (or, which is the same Thing, having equal Altitudes,) are $=$ each other; $\therefore \triangle$ DEB + $\triangle$ DBA = $\triangle$ DCB + $\triangle$ DBA; but the $\triangle$ DCB + $\triangle$ DBA = the whole quadrilateral Figure ABCD, and $\triangle$ DEB + $\triangle$ DBA = whole $\triangle$ EBA, $\therefore$ the $\triangle$ EBA = the Quadrilateral ABCD. Q. E. D.

Pl. VI. 252. PROBLEM 23. *To make a Triangle equal to*
Fig. 7. *any given five-sided Figure ABCDE.*

CONSTRUCTION. Join EC, AC, and draw DF $\parallel$ to EC, and BG $\parallel$ to CA, meeting AE produced,

duced, in the Points F and G, respectively; then draw CF and CG, and the $\triangle FCG$ will be that required.———This is so very much like the last Problem that it needs no Demonstration.

253. PROBLEM 24. *To describe a Circle through three given Points, (provided they are not in a right Line.)* Pl. VI.
Fig. 8.

CONSTRUCTION. Let A, B, C, be the three Points given; draw the Lines AB and BC, and bisect them by the Lines DE, FG, and the Point of their Intersection H will be the Center of the Circle; $\therefore$ put one Leg of the Compasses in H, and open the other Leg to one of the given Points, and with that Extent sweep a Circle, and the Circumference will pass thro' the three given Points. 226.

DEMONSTRATION. Suppose AH, BH, CH, joined; then DB being = DA, and DH common, and the contained $\angle ADH =$ the contained $\angle BDH$ by the Construction, the Side AH = Side BH; and by the same Reasoning BH = CH, $\therefore$ AH = BH = CH; consequently H is the Center of a Circle whose Circumference will pass thro' the Points A, B, and C. Q. E. D. 58.
136.

254. SCHOLIUM. *By this it is manifest, the Center of any given Circle may be found, by assuming any three Points in its Circumference, and proceeding as above. By this also, when we have a Segment of a Circle given, we may compleat the whole Circle.———This Problem also shews how to describe a Circle about a given Triangle.*

255. PROBLEM 25. *To draw a Tangent to a Circle, from a given Point D without it.* Pl. VI.
Fig. 9.

CONSTRUCTION. Let B be the Center of the Circle; join BD, which bisect in C; then about C, as 226.
as

as a Center, with the Radius CB, or its equal CD, describe a Semicircle BAD, and draw DA from D to A, the Point where the Circumference of the Semicircle intersects that of the Circle, and DA will be the Tangent required.

DEMONSTRATION. For if BA be joined, the $\angle$ BAD is a ^b Rt. $\angle$, being in a Semicircle; and $\therefore$ DA is a ^c Tangent to the Circle at the Point A. Q. E. D.

256. SCHOLIUM. If it was required to draw a Tangent from a given Point A, in the Circumference of the Circle, it would be only to erect a $\perp$ AD to the Radius BA at the Point A.

Pl. VI. 257. PROBLEM 26. Upon a given Line AB, to describe a Segment of a Circle, which shall contain a given Angle C.

CONSTRUCTION. Make the $\angle DAB$ ^a $= \angle C$; and on AD, at the Point A, erect ^c the $\perp$ AI, and bisect AB by the Line FH; then let the Point G, where the Lines AI, FH, intersect, be a Center, about which, with the Radius GA, describe the Circle ABE, and the Segment AEB will be that required.

DEMONSTRATION. Suppose B, G, joined; then AF being $=$ BF by the Construction, FG common, and contained $\angle AFG$ ^b $=$ contained $\angle BFG$, the Side AG is ^c $=$ Side BG, $\therefore$ a Circle described about G, as a Center, will pass thro' the Points A and B; let E be the Point where the Circle intersects AI, and join BE, then the $\angle AEB$ ^d $=$ BAD; but the $\angle BAD = \angle C$, by the Construction, $\therefore$ the $\angle AEB$ ^e $= \angle C$; that is, the Segment AEB contains an $\angle =$ the given $\angle C$. Q. E. D.

258. SCHOLIUM. If the given $\angle$ is a Rt. $\angle$, we have only on AB to describe a Semicircle; because the $\angle$ in a Semicircle is a ^f Rt. $\angle$.

f 148.

259. PROBLEM 27. To cut off a Segment from a given Circle ABC, which shall contain an Angle equal to a given Angle D. Pl. VI.
Fig. 11.

CONSTRUCTION. To any Point B, in the Circumference, draw ^a the Tangent EF, and make the $\angle$ FBC ^b = the given $\angle$ D; then is the Segment BAC that required. ^a 256.
^b 230.

DEMONSTRATION. For the $\angle$ FBC ^c = BAC, ^c 151. and the $\angle$ FBC also = $\angle$ D, by the Construction, $\therefore \angle$ BAC ^d = $\angle$ D. Q. E. D. ^d 45.

260. PROBLEM 28. To ^{*} inscribe in a given Circle, BCD, a Line equal to any given Line, A, less than the Diameter of the Circle.

CONSTRUCTION. About any Point B, in the Circumference, as a Center, with a Radius = the given Line A, describe the Arc *ab*, intersecting the Circumference in C; join BC, and the Thing is done. This needs no Demonstration, it being manifest from the Construction that $BC = A$. Pl. VI.
Fig. 12.

261. PROBLEM 29. In a given Circle ABC, to [†] inscribe a Triangle equiangular to a given Triangle DEF. Pl. VI.
Fig. 13.

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^{*} A Line is said to be inscribed, or placed in a Circle, when the Extremities of it are in the Circumference of the Circle.

[†] A rectilineal Figure is said to be inscribed in a Circle when all the angular Points of the inscribed Figure are upon the Circumference of the Circle.

CONSTRUCTION. To any Point A, in the Circumference, ^a draw the Tangent GAH, and make the $\angle HAC^a = \angle DEF$, and the $\angle GAB = \angle DFE$, and join BC; then is the $\triangle ABC$ that required.

^b 151. DEMONSTRATION. For $\angle HAC^b = \angle ABC$, and $\angle HAC = \angle DEF$ by the Construction, $\therefore \angle ABC^c = \angle DEF$: By the same Reasoning, $\angle ACB = DFE$. $\therefore$ the remaining $\angle BAC =$ the remaining $\angle EDF$, and so the $\triangle s$ ABC, DEF, are equiangular. Q. E. D.

Pl. VI. 262. PROBLEM 30. *About a given Circle ABC, to [†] describe a Triangle equiangular to a given Triangle DEF.*

CONSTRUCTION. Produce EF both Ways to G and H; from the Center K draw a Radius KB, and make the $\angle KBA^a = \angle DEF$, and $\angle BKC = \angle DFH$, and thro' the Points A, B, C, draw the Tangents LAM, MBN, NCL, and the $\triangle LMN$ will be that required.

DEMONSTRATION. The $\angle s$ MAK, MBK, being ^a Rt. $\angle s$, and the four $\angle s$ of any quadrilateral Figure being ^b = four Rt. $\angle s$, the $\angle AMB$ is = 2 Rt. $\angle s - \angle BKA$; but $\angle DEF^c = 2$ Rt. $\angle s - \angle DEG$, and $\angle BKA = \angle DEG$ by the Construction, $\therefore \angle AMB^d = 2$ Rt. $\angle s - \angle DEG$; $\therefore \angle AMB^e = \angle DEF$; and by the same Reasoning, $\angle BNC = \angle DFE$. Hence we have shewn that the two $\angle s$ LMN, LNM, are respectively = the two $\angle s$ DEF, DFE, and, consequently, the remaining $\angle MLN^f =$ the remaining $\angle EDF$; and $\therefore$ the $\angle s$ LMN, DEF, are equiangular. Q. E. D.

263.

[†] A rectilineal Figure is said to be described about a Circle when each Side of the circumscribing Figure touches the Circumference of the Circle.

263. PROBLEM 31. To * inscribe a Circle in a given Triangle ABC. Pl. VI.
Fig. 15.

CONSTRUCTION. Bisect ^a the $\angle$ s ABC, BCA, ^a 225.
by the Lines BI, CH, and let D be the Point of In-
tersection, and draw DF ^b $\perp$ to BC ; then about D, ^b 229.
as a Center, with a Radius = DF, describe a Circle
FGE, and it will be that required.

DEMONSTRATION. Let DE be $\perp$ to BA, and
DG $\perp$ to AC ; then the $\angle$ s BED, BFD, being
^a Rt. $\angle$ s, are = each other ; and the $\angle$ EBD ^a 14.
= $\angle$ FBD by the Construction, and the Side BD
common, $\therefore$ the Δ s EBD, FBD, are = ^b each ^b 94.
other in all Respects, and consequently ED = DF.
By the same Reason DG = DF ; $\therefore$ ED, DF, DG,
are = each other, and consequently the Circle de-
scribed about the Center D with the Radius DF,
will pass thro' the Points FEG, or, in other Words,
will touch the Sides of the given Triangle in the
Points E, F, and G. Q. E. D.

264. PROBLEM 32. To inscribe a Square in a given Circle ABCD. Pl. VII.
F. 1.

CONSTRUCTION. Through the Center E draw
the Diameter AC, and ^c $\perp$ to it the Diameter AC, ^c 226.
and join BA, AD, DC, BC, then is the rectilineal
Figure ABCD the Square required.

DEMONSTRATION. BE being = ED, AE com-
mon, and the $\angle$ AEB ^d = $\angle$ AED, the Side AB ^d 14.
will be ^e = Side AD ; and, by the same Reason, AB ^e 58.
= BC, and AD = DC ; $\therefore$ AB ^a = BC = AD = ^a 45.
DC ; that is, the rectilineal Figure ABCD is equi-
lateral. And BD being a Diameter, the $\angle$ BAD is
N 2 an

* A Circle is said to be inscribed in a rectilineal Figure when
the Circumference touches each Side of the rectilineal Figure.

- ^b 148. an $\angle$ in a Semicircle, and $\therefore$ a Rt. $\angle$: By the same Reason the $\angle$ s ABC, BCD, CDA, are each of them Rt. $\angle$ s. Hence we have shewn that the rectilineal Figure ABCD is equilateral, and its $\angle$ s
- ^c 34. all Right $\angle$ s. Consequently it is a ^c Square. Q. E. D.

Pl. VII. 265. PROBLEM 33. *To describe a Square about a given Circle ABCD.*

F. 2.

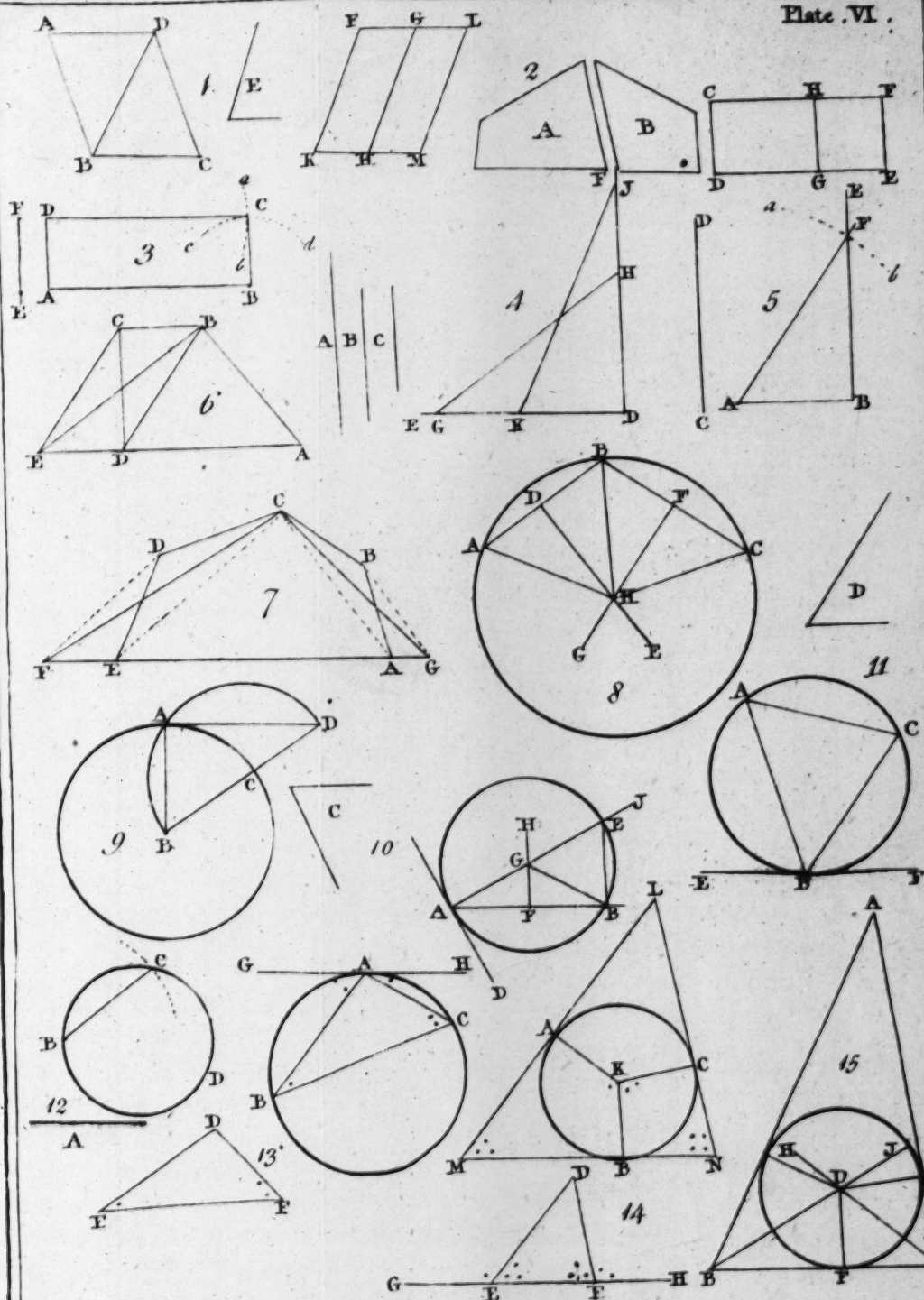
CONSTRUCTION. Draw the Diameters AC, BD, ^d 226. at Rt. $\angle$ s to each other, and thro' the Points A, B, C, D, draw the ^e 256. Tangents GF, GH, HK, KF; then will the Figure FGHK be the required Square.

DEMONSTRATION. For the $\angle$ s at A, B, C, D, are right $\angle$ s by the Construction, and because the $\angle$ s GBE, AED, are each of them Rt. $\angle$ s, they are equal to each other, and ^f 40. $\therefore$ GH ^f || to AC; ^a 74. and by the same Reason FK || to AC, $\therefore$ GH ^a || to FK. After the same Manner it may be shewn that GF is || to HK; hence the Figure FGHK is a ^a 97. Parallelogram, and its opposite Sides and $\angle$ s are equal; also GC, AK, FB, BK, are Parallelograms, and $\therefore$ their opposite Sides are equal; hence GF = BD in the Parallelogram FB, and GH = AC in the Parallelogram GC; but AC = BD being each the Diameter of the Circle, $\therefore$ GF = GH; and the opposite Sides of the Figure FGHK having been shewn above to be equal, it follows that GF = GH = HK = FK, or that it is equilateral: It is also rectangular, for the $\angle$ s GAE, AEB, EBG, being ^b 87. all Rt. $\angle$ s, the $\angle$ AGB must be also a ^b Rt. $\angle$: After the same Manner it may be shewn that the $\angle$ s at the Points H, K, F, are Rt. $\angle$ s; $\therefore$ the Figure GFKH has been proved to be both equilateral and rectangular; $\therefore$ it is a ^c 34. Square. Q. E. D.

Pl. VII. 266. PROBLEM 34. *To inscribe a Circle in a given Square FGHK.*

F. 2.

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CONSTRUCTION. ^a Bisection the Sides GF, GH, ^d 226.
by the Lines BD, AC; then about the Point of
Intersection E, with the Radius EA, sweep the
Circle ABCD, and it will be that which was re-
quired.

DEMONSTRATION. The $\angle$ s AGB, EBH, being
both Rt. $\angle$ s by the Construction, are = each other,
and $\therefore$ BE ^a || to GA; by the same Reason AE || ^a 40.
to GB; $\therefore$ the Figure AGE is a ^b Parallelogram, ^b 97.
and $\therefore$ its opposite Sides are equal, viz. AE =
GB, and BE = GA: But GB, GA, being the
Halves of equal Lines, viz. of GF, GH, (by the
Construction) are ^c = each other, $\therefore$ AE = GB = ^c 51.
BE = GA, or the Figure is equilateral; it is also
rectangular, for the $\angle$ s EAG, AGB, GBA, being
Rt. $\angle$ s by the Construction, the other $\angle$ AEB must
be a ^d Rt. $\angle$ also; and $\therefore$ the quadrangular Fi- ^d 87.
gure AGE is a ^e Square. After the same Manner ^e 34.
AFDE, DKCE, CHBE, are shewn to be Squares
= the former; $\therefore$ EA = EB = EC = ED, and
consequently, a Circle described about the Center E,
with the Radius EA, will pass thro' the Points A,
B, C, D, and ^f touch the Sides of the given Square ^f 141.
in those Points, (as the $\angle$ s at those Points are Rt. $\angle$ s.
Q. E. D.

267. PROBLEM 35. To describe a Circle about a ^{PI. VII.}
given Square ABCD. ^{F. 1.}

CONSTRUCTION. Draw the Diagonals AC, BD;
then, about the Point of Intersection E, as a Center,
with the Radius EA, describe a Circle, and it will
be that required.

DEMONSTRATION. AB being = DC, AD com-
mon, and $\angle$ BAD = $\angle$ ADC, BD ^a = AC; and ^a 58.
EA ^b = EC, and ED ^b = EB. But the whole Dia- ^b 14.
gonals AC, BD, have been just shewn to be equal,
 $\therefore$

$\therefore$ their Halves must be equal also, that is, $EA = ED = EC = EB$; and consequently a Circle described about the Point E, with a Radius $= EA$, must pass thro' the Points A, B, C, and D. Q. E. D.

268. Before we proceed any further, it may be proper to give some Definitions concerning *Polygons*.

A *Polygon* is generally defined to be a Plane Figure terminated by more than four Sides. — According to its Etymology it is a Figure having many Angles; which is in Fact the same as the above Definition; for when a Figure has many Sides it must have many Angles, *et contrà*; the Number of Sides and Angles being always equal. — When it is both *equilateral* and *equiangular*, it is called *regular*, otherwise *irregular*.

A Polygon of	$\left\{ \begin{array}{c} 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ \&c. \end{array} \right\}$	Sides is called a	$\left\{ \begin{array}{c} \text{Pentagon} \\ \text{Hexagon} \\ \text{Heptagon} \\ \text{Octagon} \\ \text{Enneagon} \\ \text{Decagon} \\ \text{Endecagon} \\ \text{Dodecagon} \\ \&c. \end{array} \right\}$
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Note. In what follows we would be understood to treat of regular Polygons only.

269. The Angle made by any two contiguous Sides of a Polygon is called the *Angle of the Polygon*. And the Angle made by two Lines drawn from the Center of the Polygon, to the extreme Points of the same Side of the Polygon, is called the *Angle at the Center*.

270. A LEMMA: *Lines bisecting the Angles of the Polygon, will intersect each other in one Point within the Polygon, and may be called Radii, for they will be equal to each other.* Pl. VII.
Fig. 3.

For the $\angle$ s of the Polygon being ^a equal, their Halves must be so too: $\therefore$ the $\angle ABS = \angle BAS$, $\therefore SA^b = SB$; and by the same Reason $SB = SC$, &c. a 268.
b 62.
 $\therefore AS = SB = SC$, &c. Q. E. D.

271. COROLLARY 1. *Hence, Lines drawn from the Center of a Polygon to the Extremities of each Side, will divide the Polygon into as many Isosceles Triangles, equal to each other in all Respects, as the Figure has Sides.*

272. COROLLARY 2. *Hence also, each Angle at the Center (as all the Angles at that Point are ^c $= 4$ Rt. $\angle$ s, or 360 Degrees,) must be $= 360$ Degrees divided by the Number of Sides.* c 63.

273. COROLLARY 3. *Hence also it follows, that the Angle of the Polygon is $= 180^\circ -$ the $\angle$ at the Center.*

For $\angle ABS + \angle SAB^a = 180^\circ - \angle$ at the Center; but $\angle ABS = \angle SAB$ by the ^b above, $\therefore (\angle ABS + \angle SAB$, or, which is the same) $2\angle ABS = 180^\circ - \angle$ at the Center; but $2\angle ABS$ is $= \angle$ of the Polygon, $\therefore$ the $\angle$ of the Polygon $= 180^\circ - \angle$ at the Center. a 78.
b 270.

274. From the two last Corollaries the following Table is easily constructed.

Name

Name of the Polygon.	Angle at Center.	Angle of the Polygon.
Pentagon	72° 00'	108° 00'
Hexagon	60 00	120 00
Heptagon	51 25 ¹ / ₇	128 34 ⁴ / ₇
Octagon	45 00	135 00
Enneagon	40 00	140 00
Decagon	36 00	144 00
Endecagon	32 43 ⁷ / ₁₁	147 16 ⁴ / ₁₁
Dodecagon	30 00	150 00

Pl. VII. 275. PROBLEM 36. *In a given Circle to inscribe a Polygon of a given Number of Sides.*

CONSTRUCTION. Draw a Radius AS, and make the $\angle ASB =$ the $\angle$ at the Center; draw AB, and it will be one of the Sides of the Polygon. Take AB in the Compasses, and set off that Distance from B to C, from C to D, &c. then will the Figure ABC, &c. be the required Polygon. This is manifest.

276. COROLLARY. *In a Hexagon, the Side of the Polygon is equal to the Radius, viz. $AB = AS$; for in that Case the $\angle$ s ASB, ABS, SAB, are each $= 60^\circ$, and $\therefore$ the $\triangle ASB$ equilateral. Note, we shall defer saying more on the Construction of Polygons till we treat of the Construction and Use of an excellent Instrument called a Sector.*

Pl. VII. 277. PROBLEM 37. *From a given Line, AB, to cut off any Part required; for Example, a Fifth of AB.*

CONSTRUCTION. Draw an indefinite Line AC, making any $\angle$ with AB; on which set off any Distance A1, set off the same from 1 to 2, from 2 to 3, from 3 to 4, from 4 to C; then have we set off
as

as many equal Parts on AC as AB is supposed to be divided into. Join B₅, and || to it, draw *a*₁, then will A*a* be $\frac{1}{5}$ Part of AB.

DEMONSTRATION. BC, *a*₁, being || to each other, as C₁ : A₁^a :: B*a* : A*a*, ∴ compoundedly C₁ + A₁ : A₁^b :: B*a* + A*a* : A*a*, that is, CA : A₁ :: BA : A*a*; consequently, A*a* is the same Part of AB as A₁ is of CA. Q. E. D.

^a 191;
^b 200.

278. SCHOLIUM. By this Problem a Line AB is divided into any Number of equal Parts. For Example Sake, A*a* being $\frac{1}{5}$ Part of AB, if we set the Distance A*a* from *a* to *b*, from *b* to *c*, and from *c* to *d*, the same would be = *dB*, and so the Line AB divided into five equal Parts. And after this Manner Scales may be constructed. But for this Purpose, *viz.* to divide a given Line into any Number of equal Parts, the following way is better adapted. For Example Sake, let it be required to divide the Line AB (see Plate VII. Fig. 20.) into five equal Parts.

CONSTRUCTION, From the Point A draw the indefinite Line AC, making any Angle at Pleasure with the given Line AB; and || to AC, draw the indefinite Line BD: Then taking any convenient Distance in your Compasses, set it from A to 1, from 1 to 2, 2 to 3, &c. also set the same Distance from B to *a*, *a* to *b*, *b* to *c*, &c. then the Line being to be divided into 5 equal Parts, join B₅, and you cannot mistake in joining the other correspondent Points, *a*₄, *b*₃, *c*₂, *d*₁; and these Lines will divide the given Line AB into 5 equal Parts, *viz.* Af = fg = gb = bi = iB.

DEMONSTRATION. It is manifest from the Construction, that the Lines *a*₄, *b*₃, &c. are all of them || to B₅, and to each other, and therefore the Δ's A₁f, A₂g, &c. are similar to each other; ∴ as

O

A₁

$A_1 : Af :: A_2 : Ag$; but if we denote A_i by m , and A_f by n , A_2 will be $= 2m$, and so the Analogy become $m : n :: 2m : Ag$; but it is manifest Ag must be $= 2n$, for $m : n$ has the same Ratio as $2m : 2n$; $\therefore Ag = 2n$, and consequently $fg = n$. Again, as $A_1 : Af :: A_3 : Ab$, that is, $m : n :: 3m : Ab$, consequently, Ab must be $= 3n$, to preserve the same Ratio: But Ag has been just shewn to be $= 2n$, $\therefore gb (= Ab - Ag$ must be) $= n$ also. After the same Manner bi and iB may be proved to be equal to n , and so $Af, fg, bi, iB =$ one and the same Quantity, and consequently equal to each other; or, in other Words, the Line AB is divided into five equal Parts, in the Points f, g, b, i , and B . Q. E. D.

N. B. We shall treat of the Construction and Use of diagonal Scales in our Essay on Trigonometry.

279. PROBLEM 38. *To find a third Proportional to two given Lines.*

Pl. VII.
Fig. 5.

CONSTRUCTION. Let AB, AC , be the two given Lines, so placed as to contain any Angle. Produce AC, AB , at Pleasure; in AB produced take $BD = AC$; join BC , and thro' D draw $DE \parallel$ to it; then will CE be the third Proportional required.

* 191.

DEMONSTRATION. For as $AB : AC :: BD : CE$; but $BD = AC$ by the Construction, $\therefore$ it is, as $AB : AC :: AC : CE$. Q. E. D.

Pl. VII.
Fig. 6.

280. PROBLEM 39. *To find a fourth Proportional to three given Lines A, B, C .*

CONSTRUCTION. Draw two indefinite Lines, Da, Db , making any Angle. In Db take $DG = A$, and $GE = B$; and in Da take $DH = C$; join GH , and draw

draw $EF \parallel$ to it; then will HF be the fourth Proportional. Q. E. I.

DEMONSTRATION. For as $DG : GE^2 :: DH^2 : HF$; or, which is the same, $A : B :: C : HF$. Q. E. D.

281. PROBLEM 40. *To find a mean Proportional to two given Lines, a, b.* PL. VII. Fig. 7.

CONSTRUCTION. Draw an indefinite Line Ad , in which take $BD = a$, $DC = b$, and upon BC describe a Semicircle; and on the Point D raise the $\perp$ DA , to cut the Circumference of the Semicircle in A , and it will be the mean Proportional required.

DEMONSTRATION. Join BA , CA , then it is manifest from Art. 210, that DA is a mean Proportional between BD and DC , or, which is the same, between the Lines a and b . Q. E. D.

N. B. This shews also how to make a Square = a given Rectangle. For if BD , DC , be the Sides of the Rectangle, DA is the Side of a Square = it: For by Art. 211, $BD \times DC = DA^2$.

282. PROBLEM 41. *To divide a Line BC according to extreme and mean Proportion.* PL. VII. Fig. 8.

CONSTRUCTION. Draw $BE \perp$ to BC and $= \frac{1}{2} BC$; and about E as a Center, with the Radius EB , describe a Circle, and thro' the Points C , E , draw the Line CA , cutting the Circumference of the Circle in the Points D and A ; then make $CF = CD$, and the given Line BC will be divided by the Point F , into extreme and mean Proportion, as was required.

DEMONSTRATION. By Art. 213, $CA \times CD = BC^2$; $\therefore BC : CA^2 :: CD : BC$, whence inversely,
O 2

ly, $CA : BC^b :: BC : CD$; this dividedly is
 $CA - BC : BC^c :: BC - CD : CD$; which in-
^c 201.
^d 198.
 versely is, $BC : CA - BC^d :: CD : BC - CD$.
 By the Construction $CF = CD$, and DA , being a
 Diameter, is = twice BE , or = BC , by the Con-
 struction; $\therefore CA = CF + BC$; hence, by writing
 CF for its equal CD , and $CF + BC$ for its equal
 CA , in the last Analogy, we have $BC : CF + BC -$
 $BC :: CF : BC - CF = BF$, or, which is the
 same, (as $+ BC - BC$ destroy each other,) $BC :$
 $CF :: CF : BF$. Q. E. D. (See Art. 176.)

283. PROBLEM 42. *To make a Square equal to any given right-line Figure.*

CONSTRUCTION. By Problem 18, find a Rect-
 angle = the given Figure; then, by Problem 40,
 find a Square = that Rectangle.

284. SCHOLIUM. When the given Figure has
 not more than five Sides, perhaps the easiest Way to
 find a Rectangle = the given Figure, is, first to
 find, by Problem 22, or 23, a Δ = that Figure;
 and then a Rectangle = that Δ may be found by
 Prob. 16.

285. PROBLEM 43. *To find two Lines which shall have the same Ratio to each other as any two given rectilineal Figures.*

CONSTRUCTION. Take two Lines at Pleasure =
 each other, and on them make Rectangles = the
 respective given Figures; then are the Rectangles
 in the same ^a Ratio as their Bases, (their Altitudes
 being equal by the Construction;) consequently, the
 given Figures must be in the same Ratio as the
 Bases of the Rectangles.

286. PROBLEM 44: Upon a given Line FG to PL. VII. describe a rectilineal Figure, similar, and similarly situated, to a given rectilineal Figure ABCDE. Fig. 9.

CONSTRUCTION. Join BE, BD, and make the $\angle GFK = \angle A$, and $\angle FGK = \angle ABE$; then make $\angle GKI = \angle BED$, and $\angle KGI = \angle EBD$. Again, make $\angle GIH = \angle BDC$, and $\angle IGH = \angle DBC$, and the Figure FGHK will be that required. After the same Manner, by proceeding from Δ to Δ , a Figure may be made similar to another of any Number of Sides.

DEMONSTRATION. The $\angle GFK$ being $= \angle BAE$, and $\angle FGK = \angle ABE$ by the Construction, the remaining $\angle FKG$ must be $=$ the remaining $\angle AEB$, and $\therefore$ the Δ s GFK, BAE, equiangular, and consequently ^a similar: After the same Manner ^b 194. the Δ GKI is shewn to be similar to the Δ BED, and the Δ GIH similar to the Δ BDC; consequently the Polygons FGHK, ABCDE, which are composed of the similar Δ s similarly situated, must be similar to each other. Q. E. D. ^a 80.

287. SCHOLIUM. When we treat of *Surveying*, we intend to give some practical Methods for solving this Problem in a more expeditious Manner.

BOOK VI.

Of the Application of Geometry to some practical Problems for measuring Heights, with the Addition of some pleasant Problems and Paradoxes for the Amusement of young Minds.

Pl. VII. 288. PROB. I. *TO find the Height of an accessible Object by Means of two Poles, of an unequal Length.*
Fig. 10.

SOLUTION. Let AB represent the $\perp$ Object, whose Height is to be found, above the Plane DB. — Let ED, the shortest Pole, be erected $\perp$ at any Point D, (which may be done by a Plumb Line, such as Masons commonly use in building Walls upright,) then, fixing your Eye at the upper End of it E, let another Person carry along the longest Pole, till you see its Top appear in a direct Line with A, the Top of the Object, where let it be stuck down; then measure the Distance DC, and CB. It is evident the Δ s EFG, EAH, are similar, and $\therefore$ we may state by the Golden Rule, as EG (or DC) : FG (the Difference of the Lengths of the two Poles) :: EH (or DB) : HA. And AH + HB (the Height of the shortest Pole) gives BA the Height of the Object. Q. E. I.

Pl. VII. 289. A LEMMA. *If a Ray of Light, EC, falls on a reflecting Surface (as a Piece of Looking-Glass) C, and is reflected to the Eye at A, the Angles ACB, ECD, are equal to each other.*
Fig. 11.

This is demonstrated by the Writers on Optics, but must here be taken for granted, till we treat of that Science.

290. PROBLEM 2. *To find the Height of an accessible Object by Means of any plain reflecting Surface, such as a Looking Glass, or Bowl of Water.*

SOLUTION. Let ED represent the Height of the Object $\perp$ to the Ground BD; at any Place on the Ground C, place the reflecting Surface; then go directly backward in the Line DC produced to B, till E the Top of the Object is seen by Reflection in the reflecting Surface at C, by the Eye at A; then measure the Height of the Eye BA, the Distances BC, and CD. Then the $\angle ACB$ being $= \angle ECD$ by the Lemma, and the Angles ABC, CDE, each Rt. $\angle$ s, the remaining $\angle BAC =$ the remaining $\angle DEC$, and so the Δ s ABC, EDC, equiangular, and $\therefore$ similar. Hence we may state by the Golden Rule, as $CB : BA :: CD : DE$, the Height of the Object required.

N. B. In our *Essay on Decimal Arithmetick*, Art. 60, is given a Method of finding the Height of an Object by its Shadow. — We shall treat more fully, and illustrate Methods more accurate in Practice, after we have treated of the Elements of Trigonometry.

291. PROBLEM 3. *To describe a Figure called an Oval.* Pl. VII.
Fig. 12.

CONSTRUCTION. Draw a Line AB, about B, with any Radius BA sweep a Circle AECK, with the same Extent of the Compasses about C, as a Center, sweep a Circle BHDI; let E and F be the Points where the Circles intersect each other; join F, B, and produce FB to G; in like Manner draw FH, EK, and EI; then setting one Leg of the Compasses in F, with the Radius FG describe the Arc GH; lastly, one Point of the Compasses being placed

placed in E, with the other describe the Arc KI; then is the Figure AGHDIK called an *Oval*.

Pl. VII. 292. PROBLEM 4. *To describe a Figure called by Fig. 13. Some an Helix, or Spiral Line.*

CONSTRUCTION. Draw a Line AB, and about any Point *a*, with any Radius *ac*, sweep the Semicircle *ced*; then take the Point *b* so that $bc = ba$, and about it, as a Center, with the Radius *bd*, describe the Semicircle *dgf*; then about the Center *a*, with the Radius *af*, describe a Semicircle *fmn*, and thus proceed at Pleasure, making the Points *a* and *b* alternately the Center of the Semicircle to be described.

Pl. VII. 293. PROBLEM 5. Paradox 1: *Having a round F. 14. & 15. Table, to make it into a square Table, and have a Hole the Middle, without Waste.*

SOLUTION. Let the round Table be cut, as represented by the cross Diameters in Figure 14, then place the four Parts together, as shewn in Figure 15; and the Thing is done.

294. PROBLEM 6. Paradox 2. *Having a round Table, to change it into two Oval-like Tables, and have a Hole in the Middle of each, without Waste of Timber.*

Pl. VII. SOLUTION. Let the Table be cut into eight F. 16 & 17. Pieces, as represented in Figure 16, then with the four Parts, 1, 2, 3, 4, may be made an Oval-like Table, as shewn in Figure 17; and with the other four Parts may be made another similar and equal to Figure 17. Q. E. I.

295. PROBLEM 7. Paradox 3. *A Gentleman having a Hole exactly a Foot square, to try the Skill of his Joiner, has ordered him to fill up that Hole with a Piece*

Piece of Board in Length 16 Inches, and Breadth 9 Inches, with a strict Charge to cut the Piece into two Parts only : How may the Joiner perform his Task ?

SOLUTION. Let the Board, Figure 18, be cut as Pl. VII. represented by the strong Lines *ab*, *bc*, *cd*, *de*, *ef* ; F. 18. & 19. then put the two Parts *Aabcde*fB and *aCDfedcb* together, as represented in Figure 19, and the Thing is done.

296. PROBLEM 8, PARADOX 4. *To describe* Pl. VIII. *Circles of different Radii, with the same Extent of the* Fig. 1. *Compasses.*

CONSTRUCTION. With any Extent of the Compasses, and one Leg in A, with the other describe a Circle : then, taking a little Solid, (suppose a Die,) resting one Leg of the Compasses on it as a Center B, with the other describe a Circle on the Paper, and it is manifest it will be of less Radius than the other.

297. PROBLEM 9, PARADOX 5. *To describe an oval-like Figure at one Sweep of the Compasses.*

CONSTRUCTION. Take a round Solid, (suppose a round Ruler,) and on it roll a Piece of Paper ; then, having opened the Compasses to any convenient Extent, fixing one Leg at any Point assumed on the Ruler, turn the other round as if you were to describe a Circle : Then take off the Paper, and the Figure so described will represent a Kind of Oval, being longer one Way than the other.

298. PROBLEM 10, PARADOX 6. *To cut a Hole in a Card, large enough for a Man to pass through.*

SOLUTION. Let ABCD represent the Card. In, Pl. VIII. or near the Middle, cut a Slit, as represented by the Fig. 2. Line *ma*, each End of it almost close to the Edge
P of

of the Card; then cut, as shewn by the Line *a1*; again, from 2 to 3, 4 to 5, 6 to 7; and thus proceed until you have cut the Line *mn*, and one Half of the Card is finished. This being done, and the other Part cut in the same Manner, on opening the Card, you will find a Hole sufficiently large for a Man to pass through. *N. B.* The nearer the Lines *a1*, 2, 3, &c. are to each other, the larger will the Hole be.

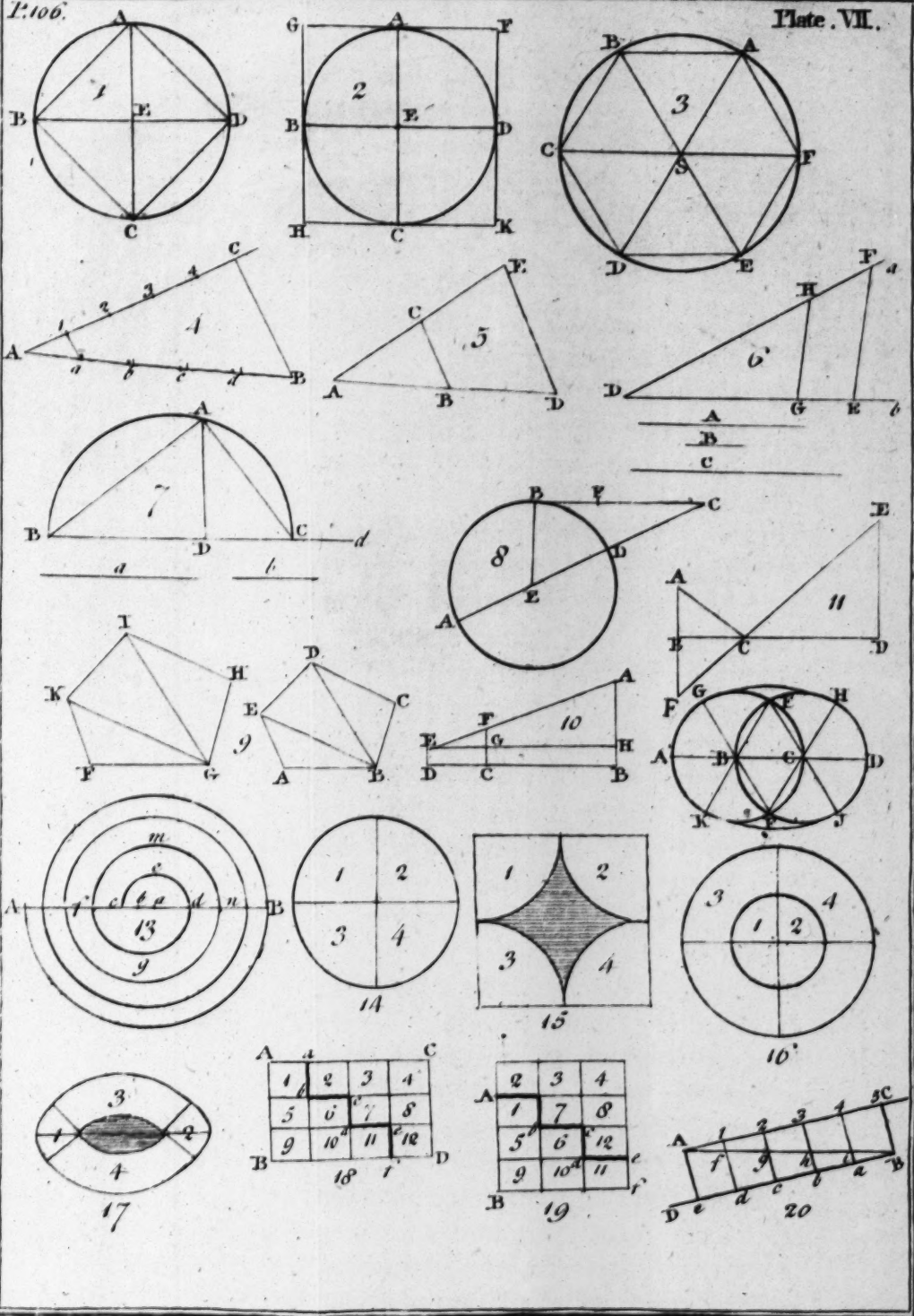
299. PROBLEM II. *To describe an Ellipsis of any given Length and Breadth.*

Pl. VIII. CONSTRUCTION. Draw the Line *AC* = the
Fig. 3. given Length, and bisect it by the Perpendicular *DB*, making *ED*, *EB*, each = $\frac{1}{2}$ the given Breadth; then is *AC* called the *Transverse Axis*, and *BD* the *Conjugate Axis*. Take *EA* or *EC* in the Compasses, and setting one Foot in *B*, with the other describe two Arcs intersecting the Transverse in the Points *F* and *G*; which Points are called the *Foci*. Then take two Pins, and placing one in the Point *A*, and the other in *C*, join them by a silk or other convenient String: This done, remove the Pins from the Points *A* and *C* to the Foci *F* and *G*; then take a Pencil in your Hand, and with it keep the String always extended, as represented at *H*; then if you begin at *A* and carry your Hand from *A* towards *D*, and from thence to *C*, the Point of the Pencil will describe the Half of the Curve *ADC*: And in the same Manner, beginning again at *A*, proceed towards *B*, and from thence to *C*, and you will have described the other Half *ABC*; and the Figure *ABCD* so described is called an Ellipsis.

Otherwise without a String, thus.

Open the Compasses to any Distance less than the transverse Axis *AC*, and with it about the Focus *F*, as a Center, describe the Arcs *ab*, *ef*; and with the same Extent of the Compasses about the Focus *G*, as a Center, the Arcs *ik*, *no*. This being done, set the





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the Distance (to which the Compasses are opened) from A towards C on the transverse Axis, *viz.* to I: Then, taking the remaining Part of the Transverse IC in the Compasses, about G as a Center, describe the Arcs *cd*, *gb*: and with the same Radius, about F as a Center, describe the Arcs *lm*, *pq*; then will the Points of their Intersections *r*, *v*, *s*, *t*, be 4 Points in the Curve of the Ellipsis. After the same Manner, by taking different Radii at Pleasure, as many Points may be found in the Curve as may be sufficient to join by an even Hand, to form the required Ellipsis.

SCHOLIUM. This Figure is one of the Sections of a Cone, and therefore is intended to be more particularly treated of hereafter; but as it is in daily Use amongst Joiners, Coopers, and other Mechanics, very few of whom can be supposed to study *Conic Sections*, we thought it would be of Service to some of them, to give the most easy Way of describing it in this Place.

300. PROBLEM 12. *To describe a Figure resembling a Hen's Egg.*

CONSTRUCTION. From a Scale of equal Parts Pl. VIII.
Fig. 4. set off 32 from A to G, and 43 from G to H. Make the $\angle AGF$ and $\angle DHC$ each $= 40^\circ$. Produce FG until GI is $= 21$ (on the same Scale of equal Parts.) Then about G as a Center, with the Radius GA describe the Arc AF; and about I as a Center, with the Radius IF, describe an Arc FE until the Chord FE is $=$ the Radius IF. Again, through the Point I draw the Line EK to intersect DH produced, in L: Then, about L as a Center, with the Radius LE describe the Arc ED. Lastly, about H as a Center, with the Radius HD describe the Arc DC, and the Curve AFEDC will represent Half of the Figure required. The other Half may be thus described, *viz.* About G as a Center de-
P 2
scribe

scribe the Arc $AM = AF$; and about H as a Center, the Arc $CP = CD$. Produce MG , and make $MN = FI$; then about N as a Center, describe the Arc MB until its Chord MB is $=$ the Radius NM . Produce BN , and make $BO = EL$; then about O as a Center describe the Arc BP ; and the Figure $ABCEA$ will be a very natural Representation of a Hen's Egg.

N. B. If the Numbers are set off from a Scale of 40 or 45 to an Inch, (in the Plate annexed we have used that of 45,) the Figure will be about the Size of a real Egg, that from which we deduced this Construction being 42 to an Inch.

SCHOLIUM. Many more Problems might have been given in this Chapter, but as they are intended chiefly for the Amusement of young Minds, perhaps these which have been already given, may by some Persons be thought more than sufficient.

A N



A N
A P P E N D I X,
CONTAINING

Some ADDITIONAL NOTES, a GEOMETRICAL SPECULUM, and an INDEX to the whole Volume.

301. **A** Note on the 6th, 7th, 8th, 9th, 10th, 11th, and 12th Articles.

In these Articles we have given the Definitions in the Order in which they stand in *Euclid*, because we set out with a Resolution to follow him as nearly as we conveniently could, consistent with our Plan; though at the same Time we cannot but acknowledge, that as they have all their Existence in a Solid, the most natural Way seemed to be, first to define a Solid, and then to shew how they exist in it; which will be extremely natural and easy, and take away the Objections commonly made against *Euclid's* Definitions, by such Persons as are not used to Abstraction: And since this Essay was sent to the Press, we find Professor *Simson* has most elegantly illustrated this, in the 2d. Edition of his *Euclid*, and therefore, as we cannot give a better, take the Illustration in his own Words, *viz.*

“ It is necessary to consider a SOLID, *that is a Magnitude which has Length, Breadth, and Thickness*, in Order to understand aright the Definitions of a Point, Line, and Superficies; for these all arise from a Solid and exist in it. *The Boundary, or Boundaries, which contain a Solid, are called SUPERFICIES, or the Boundary which is common to two Solids which are contiguous or which divides one Solid into two contiguous Parts, is called a SUPERFICIES.* Thus, if BCGF be one of the Boundaries which contain the Solid ABCDEFGH, or which is the common Boundary of this Solid and the Solid BKLCENMG, and is therefore in the one as well as the other Solid, it is called a Superficies, and has no Thickness. For if it have any, this Thickness must either be a Part of the Solid AG, or of the Solid BM, or a Part of the Thickness of each of them. It cannot be a Part of the Thickness of the Solid BM, because if this Solid be removed from the Solid AG, the Superficies BCGF, the Boundary of the Solid AG, remains still the same as it was. Nor can it be a Part of the

Thickness

Pl. VIII.
Fig. 5.

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Thickness of the Solid AG, because if this be removed from the Solid BM, the Superficies BCGF, the Boundary of the Solid BM, does nevertheless remain. Therefore the Superficies BCGF has no Thickness, but only Length and Breadth.

The Boundary of a Superficies is called a LINE, or a Line is the common Boundary of two Superficies that are contiguous, or which divides one Superficies into two contiguous Parts. Thus, if BC be one of the Boundaries which contain the Superficies ABCD, or which is the common Boundary of this Superficies and of the Superficies KBCL, which is contiguous to it, this Boundary BC is called a Line, and has no Breadth. For if it have any, this must be Part either of the Breadth of the Superficies ABCD, or of the Superficies KBCL, or Part of each of them. It is not Part of the Breadth of the Superficies KBCL, for if this Superficies be removed from the Superficies ABCD, the Line BC, which is the Boundary of the Superficies ABCD, remains the same as it was. Nor can the Breadth that BC is supposed to have be a Part of the Breadth of the Superficies ABCD, because if this be removed from the Superficies KBCL, the Line BC, which is the Boundary of the Superficies KBCL, does nevertheless remain. Therefore the Line BC has no Breadth. And because the Line BC is in a Superficies, and that Superficies has no Thickness, as was shewn; therefore a Line has neither Breadth or Thickness, but only Length.

The Boundary of a Line is called a POINT, or a Point is the common Boundary or Extremity of two Lines that are contiguous. Thus, if B be the Extremity of the Line AB, or the common Extremity of the two Lines AB, KB, this Extremity is called a Point, and has no Length. For if it have any, this Length must either be Part of the Length of the Line AB, or of the Line KB. It is not Part of the Length of KB, for if the Line KB be removed from AB, the Point B, which is the Extremity of the Line AB, remains the same as it was: Nor is it Part of the Length of the Line AB; for if AB be removed from the Line KB, the Point B, which is the Extremity of the Line KB, does nevertheless remain. Therefore the Point B has no Length. And because a Point is in a Line, and a Line has neither Breadth nor Thickness, therefore a Point has no Length, Breadth, nor Thickness. And in this Manner the Definitions of a Point, Line, and Superficies are to be understood."

302. The 8th Definition, which we have omitted, is not only *useless*, but *obscure*; and as Mr. Simson justly observes, seems to be the Addition of some less skilful Editor.

303. (Art. 57.) Having by Experience found that in printing the common Characters for *greater* and *less*, they are frequently turned the wrong Way, and so when it should be *greater* have *less*, and the contrary. In Order to remove the Confusion occasioned hereby, we have introduced the Greek G (Γ) for *greater than*, and the Greek L (Λ) for *less than*; which entirely prevent such Accidents, and at the same Time carry with them the Ideas

we.

we have Occasion to express, and therefore are no Burden to the Memory.

304. (Art. 94.) The laying of one Figure on another has been objected to as a mechanical Consideration, depending on no Postulate: But for our Parts we must acknowledge that we see no Reason why this Method of Demonstration should not be used, as its Evidence cannot be denied by any Person: And as there is no Occasion for the actual placing of one Figure on the other, but only to conceive them in the Mind to be so placed, it does not appear to us to be more mechanical than the conceiving of a Line to be drawn, or a Circle described. And *Euclid* and all other Geometricians make Use of it in demonstrating our 58th Article, which indeed does not seem capable of any other Method. As to its being grounded on no Postulate, that Objection may be easily removed, by making a fifth Postulate (after Article 44.) viz. *Grant that one Figure may be laid on another*; and certainly no Person who grants the others, will deny this, it being as manifest as either of them. Thus much we thought ourselves obliged to remark, as the Objection was made by a late learned Geometrician, for whose Memory we have the greatest Esteem.

305. (Art. 115.) As this Theorem is one of the most valuable in Geometry, it may perhaps be expected that we should give an Example or two of its Use. By this it is that any two Sides of a right-angled Triangle being given, we find the other Side: This admits of two Cases.

CASE 1. *The two short Sides being given, to find the longest.*

From the Theorem this Rule is easily deduced, viz. * The Square Root of the Sum of the Squares of the two short Sides will be equal to the longest Side.

EXAMPLE. Let DC represent Half the Breadth of a House, and be = 15 Feet, and CE the Height of the Roof = 10 Feet; it is required to find the Length of the Rafter DE.

Pl. VIII.
Fig. 10.

The Solution by the above Rule is as follows.

15	10	225	325(18
15	10	100	1
-----	-----	-----	-----
75	CE ² = 100	DC ² = 325	28)125
15	-----	-----	224
-----			-----
DE ² = 225			

Hence the Length of the Rafter is something more than 18 Feet.

After

* DEMONSTRATION. Let DC be expressed by b , CE by p , and DE by b , then by the Theorem, $b^2 + p^2 = b^2$ ∴ extracting the Square Root of each Side of the Equation, we get $\sqrt{b^2 + p^2} = b$. Q. E. D.

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After the same Manner the Side of a Square being given, its Diagonal may be found nearly true: We say nearly, because it does not admit of an exact Solution; for if we suppose the Side $= 1$, the Sum of the Squares of both Sides will be $= 2$, the Square Root of which is 1.41 nearly, but cannot be found quite accurate, and therefore, though a finite Quantity, cannot be accurately expressed in Numbers. Such Quantities are called *incommensurate*.

CASE 2. When the longest Side and one of the short Sides are given, to find the other we have this RULE. From the Square of the longest Side subtract the Square of the given short Side, and the Square Root of the Remainder will be the short Side required.*

But by Article 121. this may be worded so as to be more ready in Practice, *viz.*

Multiply the Sum of the two given Sides by their Difference, and extract the Square Root of their Product.

EXAMPLE. A Ladder 50 Feet long reached the Top of a House when its Foot was 30 Feet from the Bottom of the Wall; it is required to find the Height of the House.

The Operation will stand thus, by the first Rule.

$\begin{array}{r} 50 \\ 50 \\ \hline 2500 \end{array}$	$\begin{array}{r} 30 \\ 30 \\ \hline 900 \end{array}$	$\begin{array}{r} 2500 \\ 900 \\ \hline 1600 \end{array}$	$\begin{array}{r} 1600(40 \\ 16 \\ \hline \end{array}$
--	---	---	--

By the other Rule thus.

$\begin{array}{r} 50 \\ 30 \\ \hline \end{array}$	$\begin{array}{r} 80 \\ 20 \\ \hline \end{array}$	$\begin{array}{r} 1600(40 \\ 16 \\ \hline \end{array}$
Sum 80	1600	

Difference 20

Answer, the House is 40 Feet high.

306. (Book 2. Art. 117.) This Definition is according to *Euclid*, but every Person knows, that *two* right Lines cannot contain a Space, that is, cannot inclose any Thing, as it at least requires *three* right Lines for that Purpose; and therefore this must not be taken in a strict Sense: But when a Rectangle or right-angled Parallelogram is said to be contained by any two Lines, *Euclid* must only be understood to denote a Rectangle made on one of the Lines, and having an Altitude equal to the other.

307.

* DEMONSTRATION. By the Theorem $b^2 + p^2 = b^2$, $\therefore$ taking p^2 from both Sides of the Equation, we have $b^2 = b^2 - p^2$, $\therefore$ by extracting the

Square Root we have $b = \sqrt{b^2 - p^2}$. Q. E. D.

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307. (Book 3. Art. 144.) This Remark is also supported by Mr. *Simpson* (who is undoubtedly one of the best Judges in these Matters:) for in Page 301 of the second Edition of his *Euclid* he has this Note, *viz.* "In this we have not followed the Greek, nor the Latin Translation literally, but have given what is plainly the Meaning of the Proposition, without mentioning the Angle of the Semicircle, or that which some call the cornicular Angle, which they conceive to be made by the Circumference and the straight Line which is at right Angles to the Diameter, at its Extremity; which Angles have furnished Matter of great Debate between some of the modern Geometers, and given Occasion of deducing strange Consequences from them, which are quite avoided by the Manner in which we have expressed the Proposition. In like Manner we have given the true Meaning of PROP. 31. Book 3. without mentioning the Angles of the greater or lesser Segments. These Passages *Vieta* with good Reason suspects to be adulterated, in the 386th Page of his *Oper. Math.*"

308. (Book 4. Definition 1.) We have here to the Word *Part*, as it is in *Euclid*, prefixed *aliquot*, as it seems to be wanting to express *Euclid's* Meaning; for a lesser Magnitude may be Part of a greater, and yet not measure the greater; *e. g.* the Side of a Square is less than the Diagonal, and therefore from the Diagonal may be taken a Part = the Side; but this will not measure the Diagonal, being incommensurate to each other by Article 305.

309. We have before observed in Scholium to Article 208, that *Euclid's* Method of treating the Doctrine of Proportion is too obscure for young Students, at their first Beginning to read these Subjects; and consequently this is a sufficient Reason (as we write for Learners) for attempting to demonstrate the principal Theorems in another Manner. But as some may perhaps expect a Proof of our Assertion, that it is too difficult, it may be proper to observe, that this evidently appears from the great Disputes which have been concerning the Principles on which the *Euclidian* Method is grounded amongst the greatest Geometricians: For if they are of so difficult a Nature that the learned themselves cannot agree about them, it must follow they cannot be proper for initiating Youth, as Things cannot be well too plain for them. Mr. *Thomas Simpson*, one of the best, as well as one of the latest Writers on Geometry, thus expresses himself on this Subject in Page 274, "That there is something very ingenious and subtle in the Doctrine of Proportions as deliver'd in *Euclid's* 5th Book, cannot be denied. All that I contend for is, that the Principles on which it is built are *obscure*, and not so firmly established as to authorize its Partisans to assume that great Superiority they lay Claim to, in Point of geometrical Strictness." In Page 270 he thus remarks on the Definition which we have omitted (see our 158 Art.) *viz.* "I would seriously ask the Contemners of the vulgar and confused Notion of

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"Proportions

“ Proportions, if a Definition, by which it cannot be known
 “ whether the Ratio of the first to the second of four given Mag-
 “ nitudes may not at the same Time be both greater and less
 “ than that of the third to the fourth, is really calculated to
 “ afford those very accurate Ideas they pretend to?” And he
 even goes so far as to affirm that “ it seems sufficiently plain
 “ from *Euclid's* own Authority, that he himself was not entirely
 “ pleased with his own Performance on this Head; or that he
 “ was convinced (at least) that it had not every Advantage :
 “ For otherwise it will be very difficult to account for his having
 “ demonstrated many Things in his 7th Book, by another Me-
 “ thod, whose Demonstrations had been actually given before, in
 “ the 5th, under a different Form. For these Reasons, when I
 “ see the extravagant Commendations that have been lavished on
 “ this 5th Book of *Euclid*, I am no farther convinced by them
 “ than that great Men may sometimes launch out too far in Be-
 “ half of Opinions which they have adopted.”

Thus much seemed necessary for us to extract in our own Vin-
 dication; but as we are determined not to enter into any Dispute
 on this Subject, we shall proceed no further on this Head.

310. We shall conclude our Observations on this Book with
 the Theorem hinted at in the 116th Article, viz. *In any Triangle,*
 ABC, (the Angle at the Vertex being the greatest Angle, or at least =
 Pl. VIII. either of the Angles of the Triangle) make the Angle ACD = $\angle B$,
 F. 6, 7, & 8. and $\angle BCE = \angle A$; then will the Sum of the Squares on the two Sides
 containing the vertical Angle C, be equal to the Sum of the Rectangles
 contained by the whole Side subtending that Angle and Segments, AD,
 BE, that is, $AC^2 + BC^2 = AD \times AB + BE \times AB$.

N. B. Fig. 6. represents the Case when the $\angle C$ is obtuse,
 Fig. 7. when acute, but greater than either of the others, that is,
 when it is acute, but greater than 60 Degrees. Fig. 8. when the
 Angle is a right Angle. Fig. 9. when that Angle is not greater,
 but equal to either of the other Angles, viz. = $\frac{1}{2}$ of 2 Right
 Angles, or 60 Degrees. These Figures shew all the Varieties
 possible, and the Truth of the Theorem in all these Varieties
 will appear from one general DEMONSTRATION, viz. The Δ s
 ABC, ACD, BCE, are similar by the Construction. $\therefore$ Compar-
 ing the Triangles ACD, ABC, together, we have $AC : AD$
 • 193. $:: AB : AC$, $\therefore AC^2 = AD \times AB$. Again, by the similar
 Δ s BCE, ABC, we have $BC : BE :: AB : BC$ $\therefore BC^2 = BE$
 $\times AB$. $\therefore$ by Addition of Equals, $AC^2 + BC^2 = AD \times AB$
 $+ BE \times AB$. Q. E. D.

311. COROLLARY 1. When the Angle C is a Rt. $\angle$, the
 Fig. 8. Points D and E will coincide (as in Fig. 8.) and hence if on
 AB we make a Square, it is manifest that $AD \times AB + BE \times$
 $AB =$ the Rectangle AG + the Rectangle BG, or, which is the
 same, = the Square on AB, and $\therefore AC^2 + BC^2 = AB^2$; which
 is the famous *Pythagorean* Theorem in our 115 Article.

312. COROLLARY 2. When the $\angle C$ is $= 60^\circ$, the Δ is
 Fig. 9. known to be equiangular and equilateral, and the Points A and
 B, B and D, coincide; and so $AD \times AB + BE \times AB$ is the
 same

same as $AB^2 + AB^2$, or twice the Square on AB. But this is also known to be true from other Principles; for as the three Sides are equal, it is manifest their Squares must be so too, and consequently the Sum of any two of them must be = twice the other.

313. COROLLARY 3. From the first Corollary it evidently appears, that “ If a Line be divided into any two Parts, the Rect. Fig. 8.
angles contained by the whole and each of the Parts are together = the Square of the Whole, viz. $AG + BG = AH$, or $AB \times AD + AB \times BD = AB^2$. This is the 2d. of Euclid’s second Book.

314. COROLLARY 4. Hence also, if there be two right Lines, AB, AF, and one of them, AB, is divided into two or more Parts, the Rectangle contained by the two Lines is = the Sum of the Rectangles contained by the undivided Line AF, and the several Parts of the divided Line AB. Thus it is manifest from the Figure that the Rectangle $AG +$ the Rectangle $BG =$ Rectangle AH , or, which is the same, $AD \times AF + DB \times (BH, \text{ or its Equal}) AF = AB \times AF$.

This is the 1st. of Euclid’s second Book.

315. BOOK 5. Art. 228. It sometimes happens in Practice, that the Point C is so very near B, the End of the Line, that a sufficient Distance cannot be set from C towards B, nor the Line by some Circumstance admit of being produced longer; as for Instance, the Point C may be almost close to the Edge of the Paper; and consequently in such Case Euclid’s Method is not easy in Practice: Therefore it may be necessary to give other Methods of Construction peculiarly adapted to this PROBLEM. To draw a Perpendicular to a Line AB from any Point C, either in or very near the End B of that Line, so as to be impracticable by the Euclidian Method.

Pl. VIII.
Fig. 10.

This may be done several Ways.

1st. From a Scale of equal Parts take off 3, and set it from C to D; then from the same Scale take off 4, and placing one Point of the Compasses in C, with the other describe the Arc *ab*: Again, take off 5, and one Point of the Compasses being in D, with the other describe the Arc *cd*, and through the Point of Intersection E draw a Line CF, and it will be the Perpendicular required. For if D, E, be joined, the $\triangle DCE$ will be a Right $\angle \triangle$, right-angled at C, by Art. 305.

N. B. Any other Numbers having the same Ratio to each other, will do the same Thing; as for Example, 6, 8, and 10; or 30, 40, and 50, &c.

Or thus.

Above the Line AB assume any Point D at Pleasure, and Pl. VIII.
about it as a Center, with the Radius DC, describe a Circle (or Fig. 11.
a sufficient Part of it,) and through the Point of Intersection E, and Center D, draw the Diameter EF; then through the Points C, F, draw the Line CG, and it will be the Perpendicular required. For the $\angle ECF$ in a Semicircle is a Right Angle by Art. 148.

There are many other Methods, but these are sufficient.

316. Before we conclude our Notes on this Book, it may be of Use to some Persons to shew one of the easiest and best Methods of trying whether a Carpenter's Square is truly made; viz. apply the Square to the straight Edge of a Plank AB, as represented by the shaded Square, and by it draw the Line CD; then taking it off, turn it about, and apply it as shewn by the unshaded Square, and if, when brought to the Point C, it agrees with the Line CD before drawn, the Square is true, if not, the Difference will be double the Error of the Square.

Pl. VIII.
Fig. 12.

317. Book VI. Art. 289. It is a Maxim made Use of by some modern Philosophers, that *Nature*, or rather the God of *Nature*, does every Thing in the *shortest Way*, and indeed it is a Principle which seems to follow from the very Idea of every Thing being ordered by *Infinite Wisdom*, and as far as it has been applied, has always been found agreeable to the Nature of Things. From this Principle, by a fluxionary Process, the late Writers on Optics shew that the Angle of Incidence must be equal to the Angle of Reflection, which was the Reason of our saying, "that it must be taken for granted until we treat of that Science." But recollecting that in reasoning on *Hadley's* Octant in 1752, we thought on a very easy geometrical Way of demonstrating it, we shall here give it, with which we shall conclude our Notes.

DEMONSTRATION. Produce AB to F, making $BF = BA$, and join CF. By the Supposition $EC + CA$ is to be a *Minimum*. By the Construction $BF = BA$, the $\angle s$ ABC, FBC, each Rt. $\angle s$, and BC common, and therefore by Art. 58. the Δs are equal in all Respects, $\therefore FC = CA$. Hence, when $EC + CA$ is a *Minimum*, its Equal $EC + FC$ must be so too; but it is evident that $EC + CF$ is a *Minimum* when EF is a Right Line, the nearest Distance to any two Points being a right Line. But when EF is a Rt. Line, the $\angle ECD = \angle FCB$ by Art. 69. but the Δs ABC, FBC, have been shewn to be equal in all Respects, $\therefore \angle FCB = \angle ACB$; $\therefore$ by Axiom 1, Art. 45, $\angle ECD = \angle ACB$. Q. E. D.

SCHOLIUM. Hence it appears that Plane Geometry has in some Cases the Advantage over Fluxions, as this is very concise and evident, but the fluxionary Process something tedious and affected with Surds.

As in this Essay, for Conveniency of Demonstration, the Theorems (as in *Euclid*) relating to the same Subjects are some in one Place, some in another; and as they are in continual Use in almost every Branch of the Mathematics, we shall here collect them together in a short View, or as it may be properly called, A GEOMETRICAL SPECULUM, which will be found of the greatest Use to all young Beginners.

N. B.

N. B. The Numbers prefixed refer to the Articles in which they are demonstrated.

I. Of the Properties of Lines intersecting or making Angles with each other.

THEOREM I.

64. A Line standing on another Line making Angles with it, the Sum of those Angles is equal to two right Angles.

2.

65. If one of the Angles (in the last Article) be a right one, the other is also a right Angle ; but if one be acute, the other is obtuse.

3.

66. If several Lines stand on the same Line at the same Point, the Angles which they make are together equal to two right Angles.

4.

67. Two Lines crossing each other make four Angles, which taken together are equal to four right Angles.

5.

68. All the Angles made about a Point are $= 4$ Rt. $\angle$ s.

6.

69. If two Lines intersect each other, the opposite $\angle$ s are equal.

7.

289 If a Ray of Light, falling on a reflecting Surface, is reflected to the Eye, the $\angle$ of Incidence is $=$ the Angle of Reflection. See Art. 317.

II. Of Parallel Lines.

1.

72. If a Line crosses two Parallel Lines, the alternate $\angle$ s are $=$ each other.

2.

73. If a Line, falling upon two other Lines, makes the alternate $\angle$ s equal to each other, the Lines will be parallel.

3.

74. If 2 Lines be each of them parallel to another Line, they are parallel to each other.

4.

75. If a Line is perpendicular to another Line, it is also perpendicular to all the $\parallel$ s of that other.

5.

96. Two $\parallel$ Lines, though infinitely continued, could never meet.

III. *Of Rectangles of Lines, or Products of Quantities.*

1.
119. If a Line be divided into any two Parts, the Square on the whole Line is = the Squares on the two Parts + twice the Rectangle contained by the Parts.

2.
120. The Square on any Line is = 4 Times the Square on Half that Line.

3.
121. A Rectangle made by the Sum and Difference of two Lines is = the Difference of the Squares on the said Lines.

4.
133. If a Line be divided into any two Parts, the Rectangles contained by the Whole and each of the Parts are together = the Square of the Whole, viz. Pl. VIII. Fig. 8. $AB \times AD + AB \times BD = AB^2$.

5.
134. If there be two Rt. Lines, AB, AF, and one of them, AB, is divided into two or more Parts, the Rectangle contained by the two Lines is = the Sum of the Rectangles contained by the whole Line AF and the several Parts of the divided Line AB. Thus, $AD \times AF + BD \times AF = AB \times AF$.

IV. *Of the Properties of TRIANGLES with Respect to their Angles and Sides.*

1.
59. The $\angle$ s at the Base of an Isosceles Δ and opposite to the equal Sides are = each other; and a Line bisecting the Angle opposite the Base divides the Base into two equal Parts, and is perpendicular thereto.

2.
60. Every equilateral Δ is also equiangular.

3.
62. If 2 $\angle$ s of a Δ be equal, the Sides subtending the equal Angles will be equal.

4.
63. Every equiangular Δ is also equilateral.

5.
70. Any two Sides of a Δ are together greater than the third Side.

6.
76. If any Side of a Δ be produced, the external $\angle$ is = the 2 internal opposite $\angle$ s.

7.
77. The external $\angle$ is Γ either of the internal opposite $\angle$ s.

- 8.
78. The 3 interior $\angle$ s of every Δ are $= 2$ Rt. $\angle$ s.
- 9.
79. The 3 $\angle$ s of every Δ taken together are $=$ the 3 $\angle$ s of any other Δ taken together.
- 10.
80. If in one Δ 2 of its $\angle$ s taken together are $= 2$ $\angle$ s of another Δ taken together, the remaining $\angle$ of one will be $=$ the remaining $\angle$ of the other.
- 11.
81. If 2 Δ s have one $\angle$ of one $=$ one $\angle$ of the other, the Sum of the remaining $\angle$ s of one $=$ the Sum of the 2 remaining $\angle$ s of the other.
- 12.
82. If in any Δ one of its $\angle$ s is a Rt. $\angle$, the Sum of the other 2 is a Rt. $\angle$.
- 13.
83. If in a Rt. $\angle$ Δ one of the acute $\angle$ s is given, the other is $=$ the Difference between the given $\angle$ and a Rt. $\angle$.
- 14.
84. When the Angle contained between the equal Sides of an Isosceles Δ is a Rt. $\angle$, the other 2 $\angle$ s are each half a Rt. $\angle$.
- 15.
85. An equilateral Δ hath each $\angle = \frac{1}{3}$ of 2 Right $\angle$ s $= 60$ Degrees.
- 16.
86. If in any Δ one of its $\angle$ s is either a Rt. or an obtuse $\angle$, the other two will be each of them acute.
- 17.
89. The greater Side of every Δ is opposite to the greater $\angle$.
- 18.
90. In every Δ the greater $\angle$ is opposite to the greater Side.
- 19.
91. If two Δ s have the same common Base, the 2 Sides of the inscribed Δ taken together are less than the two Sides of the circumscribing Δ ; but the vertical $\angle$ of the inscribed greater than the vertical $\angle$ of the circumscribing Δ .
- 20.
92. If 2 Δ s have 2 Sides of one $= 2$ Sides of the other, each to each, but the contained $\angle$ of one Γ the contained $\angle$ of the other, that which has the greatest contained $\angle$, will have the greatest Base.
- 21.
93. If 2 Δ s have 2 Sides of one $= 2$ Sides of the other, each to each, but the Base of one Γ the Base of the other; the $\angle$ contained by the equal Sides of that which has the greatest Base shall be Γ the $\angle$ contained by the equal Sides of the other.
- 22.
96. In any Right $\angle$ Δ the Square upon the Side subtending the Rt. $\angle$ is $=$ both the Squares upon the Sides containing the Rt. $\angle$.

See another Demonstration in Note to Art. 211.

310. In any $\triangle ABC$ (Pl. VIII. Fig. 6, 7, 8, and 9) (the $\angle$ at the Vertex being the greatest, or at least $=$ either of the Angles of the $\triangle$) make the $\angle ACD = \angle B$, and $\angle BCE = \angle A$; then will the Sum of the Squares on the two Sides containing the vertical $\angle C$, be $=$ the Sum of the Rectangles contained by that whole Side subtending that $\angle$. and Segments AD, BE, that is, $AC^2 + BC^2 = AD \times AB + BE \times AB$.

N. B. The Theorems relating to the letting fall a Perpendicular from the vertical Angle to the Base of any Triangle, being chiefly of Use in Algebra, are referred to our intended Essay on that Art.

V. *Of the Equality of $\triangle$ s in every Respect; viz. when all their Angles and Sides are respectively equal to each other.*

1. 58. If 2 $\triangle$ s have 2 Sides of the one $=$ the 2 Sides of the other, each to each, and the $\angle$ s contained by those equal Sides equal, the $\triangle$ s will be equal in all Respects.

2. 94. If 2 $\triangle$ s have two $\angle$ s of one $=$ 2 $\angle$ s of the other, each to each, and one Side of one equal to one Side of the other, the $\triangle$ s are equal in all Respects.

3. 95. If 2 $\triangle$ s have the 3 Sides of one $=$ the 3 Sides of the other, each to each, the $\triangle$ s will be equal in all Respects.

VI. *Of the Similarity of Triangles and Proportionality of their Sides, &c.*

1. 191. If a Line be drawn $\parallel$ to one Side of a $\triangle$, it shall cut the other Sides proportional. And if the Sides, or Sides produced, are cut proportionally, the Line which joins the Points of Section shall be parallel to the remaining Side of the $\triangle$.

2. 192. If the vertical $\angle$ of a $\triangle$ be divided into 2 equal $\angle$ s by a Line cutting the Base, the Segments of the Base will have the same Ratio that the other two Sides of the $\triangle$ have to one another. And if the Segments of the Base have the same Ratio which the other Sides of the $\triangle$ have to one another, the Line drawn from the Vertex to the Point of cutting will divide the vertical Angle into 2 equal Parts.

3. 194. Equiangular $\triangle$ s are similar, that is, the Sides about the equal $\angle$ s are proportional.

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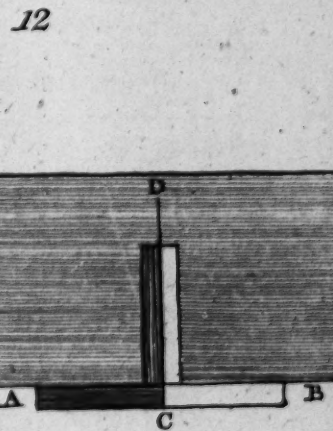
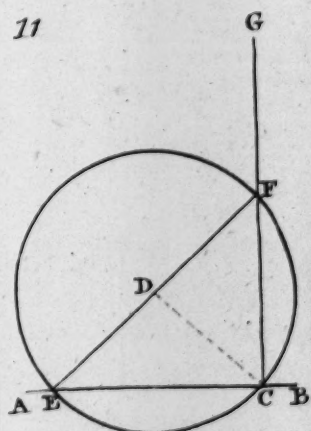
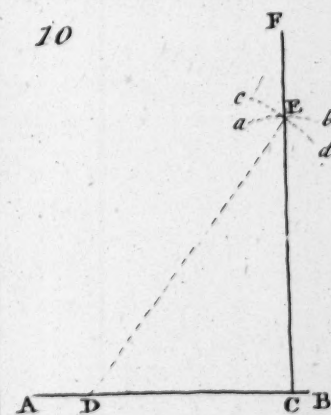
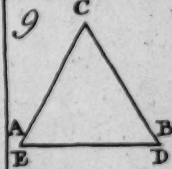
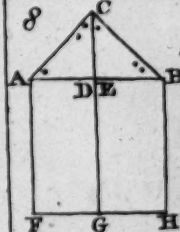
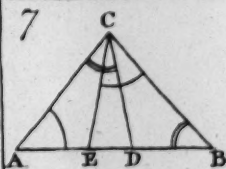
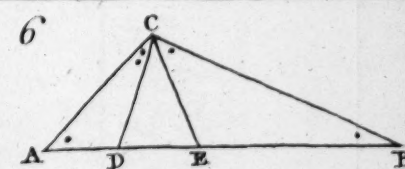
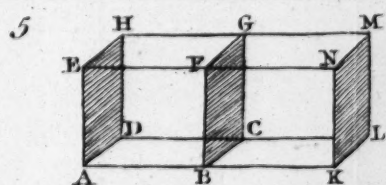
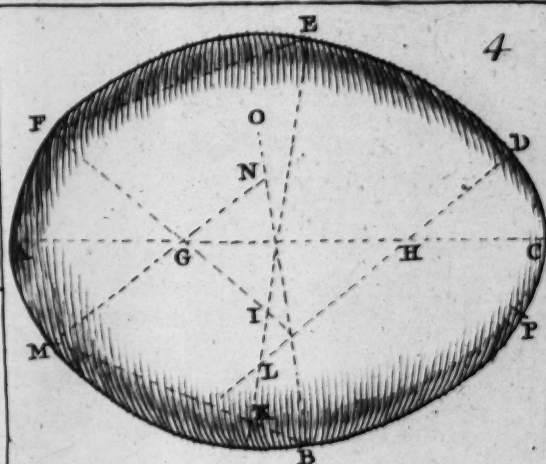
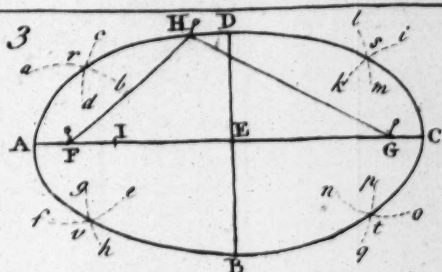
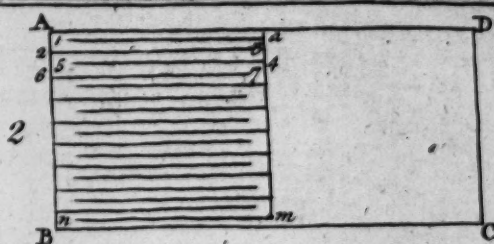
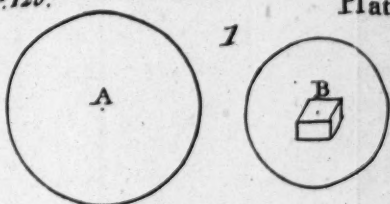
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4.

195. If the outward $\angle$ of a $\triangle$ is bisected by a Line cutting the Base produced, then the Segments between the dividing Line and the Extremities of the Base have the same Ratio which the other Sides of the $\triangle$ have to one another. And if the Segments of the Base produced have the same Ratio which the other Sides of the $\triangle$ have, the Line drawn from the Vertex to the Point of Section divides the outward $\angle$ of the $\triangle$ into 2 equal Parts.

5.

196. If 2 $\triangle$ s have one $\angle$ of one = one $\angle$ of the other, and the Sides containing the $\angle$ s proportional, the $\triangle$ s are equiangular and similar.

6.

197. If 2 $\triangle$ s have their like Sides proportional, the $\triangle$ s are similar to each other.

7.

198. If in a Rt. $\angle$ $\triangle$ a Perpendicular be let fall from the Rt. $\angle$ to the opposite Side, the $\triangle$ s on each Side of it shall be similar to the whole $\triangle$ and to each other.

VII. *Of Parallelograms, and the Comparison of $\triangle$ s and Parallelograms, with Respect to their Areas, or Equality of Surface.*

1.

97. The opposite Sides and $\angle$ s of Parallelograms are = each other : and the Diameter (or Diagonal) bisects them; that is, divides them into 2 equal Parts.

2.

99. Figures which are between the same Parallels have equal Altitudes.

3.

100. Rectangles contained under equal Sides are equal.

4.

101. If 2 Squares have the Side of one = the Side of the other ; the Squares are = each other.

5.

102. Parallelograms upon the same Base, and between the same $\parallel$ s are equal.

6.

103. Parallelograms having the same Base and equal Altitudes are equal.

7.

104. Every Parallelogram is = a Rectangle having the same Base and equal Altitudes.

8.

105. Parallelograms having equal Bases and equal Altitudes are equal.

9.

106. If a $\triangle$ and Parallelogram stand upon the same Base and are between the same Parallels, the $\triangle$ is $\frac{1}{2}$ of the Parallelogram.

R

10.

APPENDIX.

10.

107. If a Parallelogram and Δ have equal Bases and are between the same $\parallel$ s, or, which is the same Thing, have equal Altitudes, the Δ is half of the Parallelogram.

11.

108. If a Δ and Rectangle have equal Bases and Altitudes, the Δ is Half of the Rectangle.

12.

109. If 2 Δ s have equal Bases and Altitudes, they are equal to each other.

13.

111. The Area of any Parallelogram is = the Product of the Number of Parts into which the Base is supposed to be divided by the Number of like Parts contained in its Altitude.

14.

113. The Complements of Parallelograms which are about the Diagonal of any Parallelogram are = each other.

15.

114. The Diagonals of any Parallelogram bisect each other.

16.

189. Triangles and Parallelograms of the same Altitude are to one another as their Bases.

17.

190. Triangles and Parallelograms which have equal Altitudes are to one another as their Bases.

VIII. Of Polygons in general.

1.

87. All the interior $\angle$ s of any rectilineal Figure are together with 4 Rt. $\angle$ s = twice as many Rt. $\angle$ s as the Figure has Sides.

2.

88. All the exterior $\angle$ s of any rectilineal Figure are together = 4 Rt. $\angle$.

3.

98. All quadrilateral Figures, whose opposite Sides are equal, have their opposite $\angle$ s equal.

4.

114. The Diagonals of any Parallelogram bisect each other.

IX. Of regular Polygons.

1.

270. Lines bisecting the Angles of the Polygon will intersect each other in one Point within the Polygon, and may be called Radii, for they will be equal to each other.

2.

271. Lines drawn from the Center of a Polygon to the Extremities of each Side will divide the Polygon into as many Isosceles Δ s, equal to each other in all Respects, as the Figure has Sides.

272. Each $\angle$ at the Center ^{3.} is $= 360$ Degrees divided by the Number of Sides.

273. The $\angle$ of the Polygon ^{4.} is $= 1800$ — the $\angle$ at the Center. See a Table of $\angle$ s in Art. 274.

276. In a Hexagon the Side ^{5.} of the Polygon is $=$ the Radius.

X. *Of the Properties of Circles.*

^{1.}
134. If a Line drawn through the Center of a Circle bisect any Chord, it will cut it at Rt. $\angle$ s : And if the Diameter is at Rt. $\angle$ s to the Chord, it will bisect it.

^{2.}
*134. If any Point which is not the Center be taken in the Diameter of a Circle, then of all the Lines that can be drawn from that Point to the Circumference, that in which the Center is, is the longest, and the other Part of the Diameter is the least : And of the others, that which is nearer to that in which the Center is, is longer than one more distant.

^{3.}
135. If Lines drawn from any Point without a Circle to the Circumference ; then, of those which fall on the concave Part of the Circumference, the greatest will be that which passes through the Center, and the nearer to it greater than the more remote : But of those which fall on the convex Part of the Circumference, the least is that which coincides with the Diameter produced, and the nearer to it is less than the more remote.

^{4.}
136. If a Point be taken within a Circle from which there fall more than two equal Lines to the Circumference, that Point is the Center of the Circle.

^{5.}
137. One Circle cannot cut another in more than two Points.

^{6.}
139. Equal Lines in a Circle are equally distant from the Center : And those equally distant from the Center are equal to each other.

^{7.}
*139. The Diameter is the greatest Line in a Circle ; and of any other, that which is nearer to the Center is always greater than that more remote.

^{8.}
140. A Line greater than the Diameter drawn from any Point within the Circle must cut the Circumference.

^{9.}
141. A Line meeting a Circle in a Point at Rt. $\angle$ s to the Radius, is a Tangent to the Circle at that Point.

10.

142. If a Line touch a Circle in any Point, a Rt. Line drawn from the Center to that Point will be perpendicular to the touching Line.

11.

143. If a Line touches a Circle, and from the Point of Contact a Line be drawn perpendicular to the touching Line, the Center of the Circle shall be in that Line.

12.

145. The $\angle$ at the Center of a Circle is double of the $\angle$ at the Circumference upon the same Base; that is, upon the same Part of the Circumference.

13.

146. The $\angle$ s in the same Segment of a Circle are = one another.

14.

147. The opposite $\angle$ s of any quadrilateral Figure described in a Circle are together = 2 Rt. $\angle$ s.

15.

148. The $\angle$ in a Semicircle is a Rt. $\angle$; but the $\angle$ in a Segment greater than a Semicircle is less than a Rt. $\angle$; and the $\angle$ in a Segment less than a Semicircle is greater than a Rt. $\angle$.

16.

149. In an oblique Δ inscribed in a Circle, the vertical $\angle$ exceeds, or is less than a Rt. $\angle$; by the $\angle$ made by the Base and the Diameter drawn from either Extremity of the Base.

17.

150. In equal Circles the Angles at the Circumference, and standing upon equal Chords, are = each other: And if the Circumferences are equal, the Chords on which they stand will be equal also.

18.

151. If a Line touches a Circle, and from the Point of Contact a Line be drawn cutting the Circle, the $\angle$ s made by this Line with the Line touching the Circle, shall be = the $\angle$ s which are in the alternate Segment of the Circle.

19.

210. (See Pl. IV. Fig. 13.) 1st. As $AB : BC :: AD : AC$; 2d. As $BD : DA :: DA : DC$ (this is all we mean when we say AD is a geometric mean Proportional between BD and DC.

20.

211. Therefore 1st. $AB \times AC = BC \times AD$; 2d. $BD \times DC = DA^2$; 3d. $BC \times BD = AB^2$; 4th. $BC \times DC = AC^2$.

21.

212. If 2 Lines cut one another within a Circle, (as in Plate IV. Fig. 14.) the Rectangle of the Segments of one of them is = the Rectangle of the Segments of the other, viz. $AE \times EC = BE \times ED$. And if two Lines cutting a Circle, being produced, intersect each other without the Circle, (as in Pl. IV. Fig. 15.) then also will $AE \times EC = BE \times ED$.

22.

213. If from any Point C without a Circle, (Pl. IV. Fig. 16.) 2 Lines CA, CB, are drawn in such a Manner that CB touches the Circle, and CA cuts it, then will $CA \times CD = BC^2$.

23.

214. In equal Circles, $\angle s$, whether at the Centers or Circumferences, have the same Ratio with the Circumferences on which they stand. Thus, let G and H (Pl. IV. F. 17.) be the Centers of the two Circles; then we affirm, that, as the Circumference BC : the Circumference EF :: the $\angle BGC$: the $\angle EHF$; and :: $\angle BAC$: $\angle EDF$.

24.

215. The $\angle BGC$ in the Center : 4 Rt. $\angle s$:: the Circumference or Arc BC on which it stands : the whole Circumference of the Circle.

25.

216. Hence the Circumferences DE, BC, (Pl. IV. Fig. 18.) of unequal Circles, which subtend equal Angles at the Centers, as $\angle s$ DAE, BAC, are similar; viz. As Arc DE : the whole Circumference of the Circle DEF :: BC : the Circumference of the Circle BCG. And if they are similar, they will subtend equal Angles.

N. B. As to the Theorems relating to the Circumferences of Circles being as their Radii or Diameters. The Areas of similar Plane Figures being as the Squares of their like Sides. &c. we intend to give them in a more proper Place, viz. when we treat on *Mensuration*.

XI. Of Proportion.

1.

187. Magnitudes have the same Ratio to one another which their Equimultiples have.

2.

188. If 4 Quantities are proportional, then any Equimultiples of the 2 first, will have the same Ratio as any Equimultiples of the two last.

3.

193. If 4 Lines are proportional, the Rectangle contained by the Extremes is = the Rectangle contained by the Means.

4.

198. If 4 Quantities are proportional, they will also be Proportionals taken inversely, viz. if $a : b :: c : d$; then will $b : a :: d : c$.

5.

199, 200, 201, 202, 203. If 4 Quantities are proportional, they will also be proportional when taken alternately, compoundedly, by Division, by Conversion, and mixtly, viz. if $a : b :: c : d$; then,

199. $a : c :: b : d$. Alternately.

200. $a + b : b :: c + d : d$. Compoundedly.

201. $a - b : b :: c - d : d$. By Division.

202.

APPENDIX.

202. $a : a - b :: c : c - d$. By Conversion.

203. $a + b : a - b :: c + d : c - d$. Mixtly.

6.

205. If 3 Lines are proportional, the Rectangle under the Extremes is = the Square on the Mean, viz. if $a : b :: b : c$, then $ac = b^2$.

7.

206. The Rectangles under proportional Lines are proportional. Thus, if $a : b :: c : d$, and $e : f :: g : h$; then $a \times e : b \times f :: c \times g : d \times h$.

8.

207. The Squares on 4 proportional Lines are Proportionals, viz. if $a : b :: c : d$; then will $a^2 : b^2 :: c^2 : d^2$.

9.

208. If 4 Squares are Proportionals, their Sides will be so too.

A LIST of the PROBLEMS.

PROBLEM I.

111 and 112. To find the Areas of Parallelograms and Triangles.

2.

221. To draw a Line from a given Point = a given Line.

3.

222. From a greater Line to cut off a Part = a lesser Line.

4.

223. About a given Point to describe a Circle whose Radius shall be = a given Line.

5.

224. To make a Δ whose Sides shall be respectively = three given Lines.

6.

225. To bisect a given $\angle$, that is, to divide it into two equal Parts.

7.

226. To bisect a given Line.

8.

227. The last Art. serves also to raise a Perpendicular on the Middle of a Line.

9.

228. To draw a Perpendicular to a given Line from any Point in it. See also Art. 315.

10.

229. To let fall a Perpendicular from a Point above to a Line below.

11.

230. To make an Angle = a given $\angle$.

12.

231. To construct and shew the Use of a Protractor.

234. To construct and shew the Use of a Line of Chords. 13.
236. To draw a Line parallel to a given Line through a given Point. 14.
237. At any Distance to draw a Line $\parallel$ to another Line. 15.
239. To construct a parallel Ruler. 16.
241. To trisect an Angle. 17.
242. To describe a Parallelogram that shall be $=$ a given Δ , and have one of its $\angle$ s $=$ a given $\angle$. 18.
243. To apply a Parallelogram to a given Line, which shall be $=$ a given Δ , and have one of its $\angle$ s $=$ a given $\angle$. 19.
244. To describe a Parallelogram $=$ a given rectilineal Figure, and having an $\angle =$ a given rectilineal $\angle$. 20.
246. By this also may be described a Parallelogram $=$ the Sum, or Difference, of any two given right-lined Figures. 21.
247. To describe a Rectangle whose Sides shall be $=$ two given Lines. 22.
248. Hence also to construct a Square. 23.
249. To make a Square $=$ any Number of given Squares. 24.
250. To make a Square $=$ the Difference of two given Squares. 25.
251. To make a $\Delta =$ any given quadrilateral Figure. 26.
252. To make a $\Delta =$ any given 5-sided Figure, 27.
253. To describe a Circle through 3 given Points. 28.
254. To find the Center of a Circle. — By having a Segment of a Circle to compleat the Circle. — And to describe a Circle about a given Δ . 29.
255. To draw a Tangent to a Circle from a given Point without it. 30.
256. To draw a Tangent from a given Point in the Circumference of the Circle. 31.
257. Upon a given Line to describe a Segment of a Circle which shall contain a given Angle. 32.

33.
259. To cut off a Segment from a given Circle which shall contain a given $\angle$.

34.
260. To inscribe in a given Circle a Line = any given Line, less than the Diameter of the Circle.

35.
261. In a given Circle to inscribe a Δ equiangular to a given Δ .

36.
262. About a given Circle to describe a Δ equiangular to a given Δ .

37.
263. To inscribe a Circle in a given Δ .

38.
264. To inscribe a Square in a given Circle.

39.
265. To describe a Square about a given Circle.

40.
266. To inscribe a Circle in a given Square.

41.
267. To describe a Circle about a given Square.

42.
275. In a given Circle to inscribe a Polygon of a given Number of Sides.

43.
277. From a given Line to cut off any aliquot Part.

44.
278. To divide a given Line into any Number of equal Parts. Being of Use in making Scales of equal Parts.

45.
279. To find a third Proportional to two given Lines.

46.
280. To find a fourth Proportional to three given Lines.

47.
281. To find a mean Proportional to two given Lines.

48.
282. To divide a Line according to extreme and mean Proportion.

49.
283. To make a Square = any given right-line Figure.

50.
285. To find two Lines which shall have the same Ratio to each other as any two given rectilineal Figures.

51.
286. Upon a given Line to describe a rectilineal Figure, similar, and similarly situated, to a given rectilineal Figure.

52.
288. To find the Height of an accessible Object by Means of two Poles of unequal Length.

290. To find the Height of an accessible Object by Means of any plane Reflecting Surface, as a Looking-Glass, or Bowl of Water.

291. To describe a Figure called an Oval.

292. To describe a Figure called by some an Helix, or Spiral Line.

293. Having a round Table, to make it into a square Table, and have a Hole in the Middle, without Waste.

294. Having a round Table to change it into two oval-like Tables, and have a Hole in the Middle of each, without Waste of Timber.

295. To fill a Hole exactly a Foot square, with a Piece of Board in Length 16 Inches, and Breadth 9 Inches, by only cutting it into two Parts.

296. To describe Circles of different Radii with the same Extent of the Compasses.

297. To describe an Oval-like Figure at one Sweep of the Compasses.

298. To cut a Hole in a Card large enough for a Man to pass through.

299. To describe an Ellipsis of any given Length and Breadth.

300. To describe a Figure resembling a Hen's Egg.

305. Any two Sides of a right-angled $\triangle$ being given, to find the other Side.

As many valuable Authors in their Works frequently refer to some Propositions of *Euclid*, we have added the following Tables, to shew by Inspection what Articles in this Essay answer to the most useful Theorems in that Author, and the contrary.

APPENDIX.

A TABLE shewing by Inspection what Article in this ESSAY answers to its corresponding Proposition in *Euclid*.

Propo- sitions in <i>Eu- clid</i> .	Articles in this Essay.	Propo- sitions in <i>Eu- clid</i> .	Articles in this Essay.	Propo- sitions in <i>Eu- clid</i> .	Articles in this Essay.	Propo- sitions in <i>Eu- clid</i> .	Articles in this Essay.
Book I. Prop.		Book II Prop.		Bo. III. Prop.		Bo. VI. Prop.	
2	221	1	314	1	260	1	189
3	222	2	313	2	261	2	191
4	58	4	119	3	262	3	192
6	62	5	121	4	263	4	194
8	95	11	282	5	254	5	197
9	225	14	283	6	264	6	196
10	226			7	265	8	209
11	228	Bo. III. Prop.		8	266	9	277
12	229	2	254	9	267	10	278
13	64	3	134	10	relate to de- scrib- ing Po- lygons see Ar- ticle 275.	11	279
15	69	7	*134	11		12	280
18	89	8	135	12		13	281
19	90	9	136	13		16	193
20	70	10	137	14		17	205
21	91	14	*138	15		18	286
22	224	15	139	16		33	214
23	230	16	141				
24	92			Book V Prop.			
25	93	17	{ 255	4	188		
26	94	18	{ 256	15	187		
27	73	19	142	16	199		
29	72	20	143	17	201		
30	74	21	145	18	200		
31	236	22	146				
32	76	25	147				
32	78	30	254				
1C. 32	87	31	225				
2C. 32	88	32	148				
34	97	33	151				
35	102	34	257				
36	105	35	259				
41	106	36	212				
42	242		213				
43	113						
44	243						
45	244						
46	248						
47	115						

APPENDIX.

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A TABLE shewing what Proposition in *Euclid* answers to its corresponding Article in this ESSAY.

Art. in this Essay.	Propos. in <i>Euclid</i> .		Art. in this Essay.	Propos. in <i>Euclid</i> .		Art. in this Essay.	Propos. in <i>Euclid</i> .		Art. in this Essay.	Propos. in <i>Euclid</i> .	
	Prop.	Book		Prop.	Book		Prop.	Book		Prop.	Book
58	4.	1	119	4.	2	199	16.	5	261	2.	4
59	5.	1	121	5.	2	200	18.	5	262	3.	4
62	6.	1	134	3.	3	201	17.	5	263	4.	4
64	13.	1	* 134	7.	3	205	17.	6	264	6.	4
69	15.	1	135	8.	3	209	8.	6	265	7.	4
70	20.	1	136	9.	3	212	35.	3	266	8.	4
72	29.	1	137	10.	3	213	36.	3	267	9.	4
73	27.	1	* 138	14.	3	214	33.	6	275 shews how to describe Po- lygons, which <i>Euclid</i> does in 10, 11, 12, 13, 14, 15, 16, of 4th Book, but in a Method entirely differ- ent.		
74	30.	1	139	15.	3	221	2.	1			
76	32.	1	141	16.	3	222	3.	1			
78	32.	1	142	18.	3	224	22.	1			
87	1C. 32. 1.		143	19.	3	225	9.	1			
88	2C. 32. 1.		145	20.	3	226	10.	1			
89	18.	1	146	21.	3	228	11.	1			
90	19.	1	147	22.	3	229	12.	1			
91	21.	1	148	31.	3	230	23.	1			
92	24.	1	151	32.	3	236	31.	1			
93	25.	1	187	15.	5	242	42.	1	277	9.	6
94	26.	1	188	4.	5	243	44.	1	278	10.	6
95	8.	1	189	1.	6	244	45.	1	279	11.	6
97	34.	1	191	2.	6	248	46.	1	280	12.	6
102	35.	1	192	3.	6	254	2.	3	281	13.	6
105	36.	1	193	16.	6		5.	4	282	11.	6
106	41.	1	194	4.	6	255	17.	3	283	14.	2
113	43.	1	196	6.	6	256			286	18.	6
115	47.	1	197	5.	6	257	33.	3	313	2.	2
						259	34.	3	314	1.	2
						260	1.	4			

The END.

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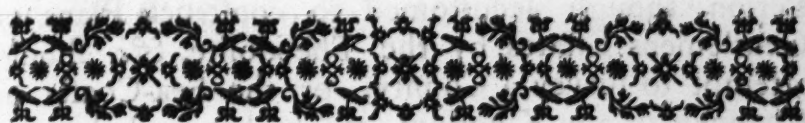
AN
E S S A Y
ON THE
E L E M E N T S
O F

Plane Trigonometry.

With their APPLICATION to ALTIMETRY
and LONGIMETRY.

BY BENJAMIN DONN.

*Trigonometria plana est scientia ex tribus trianguli rectilinei partibus
inveniendi reliquas.*



The P R E F A C E.

HA V I N G already given the Elements of Geometry, we come now to treat of Plane Trigonometry.

It is by Trigonometry that we apply Geometry to find the Heights of Objects and Distances of Places, both on Sea and Land, as is shewn in this Essay. Hence it appears, that, even in the narrow Limits to which it is here confined, it is a Science of great Use in the common Affairs of Life; and, if we include the Doctrine of spherical Triangles also, under the Term Trigonometry, it is that Science to which may be attributed whatever is effected in Astronomy, Geography, and Navigation, agreeably to the Motto prefixed to this Volume. For by Trigonometry it is the Astronomer computes the Distances of the Planets, calculates the Eclipses of the Sun and Moon, the Transits of Venus and Mercury, constructs Dials for shewing the true Time of the Day, &c. By it the Geographer lays down the principal Places on the terraqueous Globe in their true Positions and Distances: And, lastly, by it the Navigator conducts his Ship, through the trackless Ocean, to his desired Port, though ever so far distant, by the uncertain Impulse of the Wind.

What we have attempted to do, in this Essay, is, to give the Elements of Plane Trigonometry in a clear Manner, with their Application to Altimetry and Longimetry, in a greater Variety of useful Problems, both for Land and Maritime Surveying, than is given in any other Treatise on this Subject.

P R E F A C E.

And here it may not be improper to hint, that no Person, though accustomed to construct Plans of Gentlemens Estates by the Theodolite, &c. should attempt to survey a County, or the Sea-Coasts or Channels of any Kingdom, till he has made himself Master of these Problems, with as much of Navigation or Astronomy, at least, as teaches to find the Latitudes and Longitudes of Places; which done, for his Encouragement, we will venture to affirm there will not arise any Difficulty, in Practice, which he may not easily surmount.

The Writers on this subject are numerous; as almost all Authors, who have treated of Surveying, Navigation, &c. have written preparatory Treatises on Plane Trigonometry: Some of them are, *Pitiscus*, in the Year 1614; *Gunter*, 1623; *Briggs*, 1633; *Newton*, 1658; *Sturmy*, 1667; *Norwood*, 1631; *Seller*, 1669; *Dechales*, 1674; *Phillipps*, 1678; Sir *Johnas Moore*, 1681; *Atkinson*, 1686; *Leybourn*, 1690; *M. Bouger*, 1694; *Leybourn*, 1704; *Sherwin*, 1705; *Harris*, 1706; *Hodgson*, 1706; *Ozanam*, 1712; *Laurence*, 1717; *Wilson*, 1720; *Hawney*, 1725; *Keil*, 1728; *Wolfius*, 1732; *Kelly*, 1734; *Martin*, 1736; *Hauxley*, 1743; *Simpson*, 1748; *Muller*, 1748; *Emerson*, 1749; *Barrow*, 1750; *Webster*, 1751; *M. Bouger*, 1753; *Holliday*, 1756; *Martin*, 1759; *Maseres*, 1760; *Mountaine*, 1763; *Robertson*, 1764; *Patoun*, 1765; *Simson*, 1767; *Hutton*, 1770; *Payne*, 1772.



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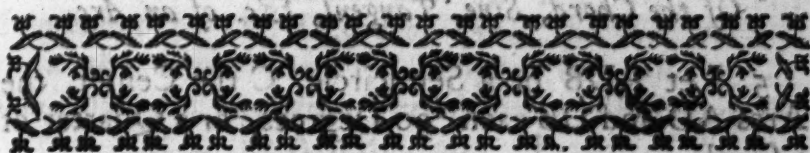
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THE
ELEMENTS
OF
Plane Trigonometry.

BOOK I. CHAP. I.

Of DEFINITIONS, and the Nature of SINES, TANGENTS, &c.

1. **T**RIGONOMETRY is the Science or Doctrine of the Properties of Triangles.

2. A *Triangle* is a Figure contained under three Lines.

3. Trigonometry may be either *Plane* or *Spherical*. *Plane Trigonometry* is, when the three Lines which form the Triangle are Right Lines. It is called *Plane*, because these Lines are in the same Plane, or flat Surface. By some Authors it is called *Plain*, because the common Principles of Plane are Something easier to be understood than those of Spherical Trigonometry.

4. *Spherical Trigonometry* treats of the Properties of Triangles, formed, or conceived to be formed, on the Superficies of a Globe or Sphere; which are therefore called *Spherical Triangles*. But this must be the Subject of some future Essay.

Of the Chord, Sine, Tangent, &c. of an Arc.

Pl. I. Fig. 1, 2. 5. Let ADB be a Semicircle, C the Center, the Arcs AD and DB each $^{\circ}90$ Degrees, or a Quarter of a Circle.

First. Assume any Point, F, in the Arc BD; join B, F; then the Line BF is called the *Chord* of the Arc BF.

6. *Secondly.* From the Point F, let fall the Line FE, perpendicular to AB; or (which is the same) draw FE parallel to DC; then is the Line FE called the *Sine*, and EB the *Versed-Sine*, of the Arc BF.

7. *Thirdly.* At the Point B, on AC, raise the Perpendicular BG; join the Points, C, F, by the Line CF; which produce, to intersect BG in G; then is BG called the *Tangent*, and CG the *Secant*, of the Arc BF.

8. *Fourthly.* Any Arc, BF, less than a Quarter of a Circle, being subtracted from 90 Degrees, or the Quarter of the Circle DB, the remaining Arc FD is called the *Complement* of the Arc BF.

9. *Fifthly.* From the Point, F, draw FI, perpendicular to CD; which is easiest done by drawing it parallel to AB: Also, from the Point D, DH parallel to AB, (or, which is the same, perpendicular to DC,) till it intersects CG in H: Then is IF the *Sine-Complement*, DH the *Tangent-Complement*, and CH the *Secant-Complement*, of the Arc FB; or (which is the same Thing) the *Sine*, *Tangent*, and *Secant*, of the Arc DF.

10. Instead of Sine-Complement, Tangent-Complement, and Secant-Complement, it is very common to write Co-sine, Co-tangent, and Co-secant.

11. By the Construction it appears that CE is always equal to IF, the *Cofine* of the Arc BF.

12. Hence

* The Reference, 217. g. at this Mark in the Margin, signifies, Article 217, Elements of Geometry: And, in general, when g. is added to a Reference, it denotes the Elements of Geometry; otherwise, the Article in this Essay.

Explanation of Characters.

3

12. Hence it appears, that any Arc, BF, and its Supplement to 180 Degrees, (*viz.* the Arc ADF,) have the same Sine FE, Tangent BG, and Secant CG, but not the same Versed-Sine; BE being the Versed-Sine of the Arc BF, and AE that of the Arc ADF.

13. It appears, from Figure II. that, when the Arc BF is equal to 45 Degrees, the Radius CB is equal to the Tangent BG.

For, as the Triangle CBG is right-angled at B, and the $\angle$ BCG being 45 Degrees by the Hypothesis, the Angle CGB is also 45 Degrees; and therefore, the Sides, CB, BG, being opposite to equal Angles, must be equal. 83. g.

14. As Arcs are the Measures of Angles, it is as customary to say, the Sine, Tangent, or Secant, of an Angle, as the Sine, Tangent, &c. of an Arc. 62. g.

15. Before we proceed any farther, it may be proper to insert the Explanation of such Characters as will be used in this Essay, which are not already explained in our *Elements of Geometry*. Viz.

CHARACTERS.

S.	Sine.
S. C.	Sine-Complement.
C. S.	Cofine.
T.	Tangent.
T. C.	Tangent-Complement.
Sec.	Secant.
V. S.	Versed-Sine.
Sup.	Supplement to 180 Degrees.
d. or °	Degrees } as 3° 17' 14" is to be read 3 De-
'	Minutes } grees, 17 Minutes, 14 Se-
"	Seconds } conds.
h.	Hours, as 5h. 10' is 5 Hours, 10 Mi-
	nutes.

16. Before we proceed to the Solutions of the common Cases of Plane Triangles, it is proper to explain the Construction of the several Lines gene-

Construction of Scales.

rally put on a Plotting-Scale, viz. Lines or Scales of equal Parts, also of Chords, Sines, Tangents, and Secants, for the more readily constructing Triangles, &c.

B O O K I. C H A P. II.

Of the Construction of Scales.

17. **A** SCALE of 10 equal Parts in an Inch, as AD, may be constructed by a few Trials (or by Art. 278, of *Geometry*).

18. If it be required to make any other Scale of equal Parts, (as, for Example, 15 to an Inch,) it may be done thus.

Pl. I. Fig. 3. First, Draw AC $\perp$ to AB, and from any convenient Place, C, draw CF, making any convenient Angle, at Pleasure, with the Line CA; join C, D, and make CI = AD; then take from the Scale, AB, the Number of Parts into which the new Scale is to be divided; viz. in this Example, extend the Compasses from E to a, and set that Extent from C on the Line CA (produced if necessary) to K, and join IK, and draw AL $\parallel$ to KI; make CM = CL, and draw MN $\parallel$ AB; then will Mb be = one of the primary Divisions, on the New Scale.*

Then

* DEMONSTRATION. Since the Number of Divisions is to be greater in an Inch Length, in the Scale MN, than in the Scale AB, it is manifest that the primary Divisions must be decreased in that Proportion; that is, AD must be, in this Example, to Mb, as 15 to 10.

That it is so, in the above Construction, will easily appear; for CK was taken to CI in that Proportion; and, AL being $\parallel$ to IK, the Δ s CIK and CLA are similar; $\therefore$ CL has the same Ratio to CA as CI to CK. But CM is made = CL; $\therefore$ CM : CA :: CI : CK. But, Mb being $\parallel$ AD, the Δ s CMb and CAD are similar. $\therefore$ CM : CA :: Mb : AD; $\therefore$ by Equality, Mb : AD as CI to CK, the given Proportion. Q. E. D.

After this, that Mb, by drawing Lines from C to the several Divisions in AD, will be divided into 10 equal Parts, is too manifest to need any Explanation.

Construction of Scales.

Then Lines drawn from the Point C, to the several Divisions in AD, will divide Mb into 10 equal Parts.

19. The Use of a Scale of equal Parts, as represented by AB.

The primary Division 1 may be considered either as $\frac{1}{10}$, 1, or 10, or 100, &c. and then the Division 2 must be reckoned proportional; as, if 1 is $\frac{1}{10}$, 2 is $\frac{2}{10}$; if 1 be 1, 2 is 2; if 1 stands for 10, 2 is for 20; if 1 for 100, 2 for 200, &c. And as the Divisions on the Left, viz. AD, are divided into 10 equal Parts, each is $\frac{1}{10}$ of what the first primary Division, DE, is supposed to be.

For Example. Let it be required to take off $\frac{3}{10}$ from the Scale, supposing each primary Division to be 1. Place one Point of the Compasses on D; then open the other to the fifth Division at a; and the Compasses will be opened to $\frac{5}{10}$.

Again: Let it be required to take off 25, supposing the Division at E to denote 10. Extend the Compasses from B to a, and it will be the Distance required.

The same Extent, if 1 at E be considered as 100, will denote 250.

20. But, when it is required to take off a Number expressed by three or more Figures pretty accurately, it is more convenient to have a *Diagonal Scale*; which is thus constructed.

For Example's Sake, let it be required to construct a Scale for dividing an Inch into 100 equal Parts.

Having drawn *ab*, *cd*, and made the primary Divisions, consider how many Parts you would have a Pl. I. Fig primary Division, *ae*, divided into; suppose, as in 4. the common Scales, 100; then the Number of Divisions in *ce*, $\times$ by the Number of Divisions in *cm*, must be equal to 100; and $\therefore$ each is generally made

Construction of Scales.

made to contain 10 Divisions, 10 Times 10 being 100. Through the several Divisions in *cm*, and parallel to *cd*, draw the 10 parallel Lines, as represented in the Figure : Divide *ce* and *mn* each into 10 equal Parts, and join the Points by the Diagonal Lines, as shewn in the Figure.

21. An Example or two will explain the Use of this Scale.

First, suppose it is required to take off 156 in the Compasses, the primary Division being 100.

Put one Point of the Compasses at the Place where the 6th || Line intersects the vertical Line of the first primary Division, *viz.* at *o*, and extend the other Leg to the Place where the 5th Diagonal Line cuts the same Parallel ; and the Thing is done.

22. EXAMPLE 2. Suppose it is required to take off 2740, each primary Division being 1000.

Place one Leg of the Compasses at the Place where the 4th Parallel intersects the 2d primary Division, *viz.* at *r*, and extend the other on the same Parallel, to the 7th Diagonal ; and the Extent of the Compasses will be the Distance required.

Hence the Learner may observe, that the parallel Divisions in the Diagonal Scale are considered but as $\frac{1}{10}$ of the diagonal Divisions.

23. *Scholium.* For other Purposes, the Diagonal Scale must be otherwise divided.

For Example, if to plot Distances taken by a Perambulator, or Measuring-Wheel, which takes Miles, Half-Miles, and Poles, (which I have found to be best for surveying Counties,) then a Mile containing 320 Poles, I construct it thus : Make *ae* = $\frac{1}{2}$ a primary Division, or Half a Mile ; then divide it into 16 equal Parts, and *cm* into 10 ; for $16 \times 10 = 160 = \frac{1}{2}$ a Mile.

But if the Wheel takes Miles, Furlongs, and Poles, as some are made, (which I would not recommend, for Reasons which will appear in a more proper Place,) then I divide each primary Division

into

Construction of Scales.

into 8 Parts, for Furlongs, and draw vertical Lines, as in the primary Divisions; after which I make ce = the Distance for a Furlong, and divide it into 4 equal Parts, and cm into 10; because $4 \times 10 = 40$ Poles = 1 Furlong.

In my Survey of *Devon*, I had both these Kinds of Wheels; which occasioned my constructing Scales on these two Methods; and, as young Surveyors might perhaps be at some Loss to contrive them, I take this Opportunity to hint at them here, rather than wait till our Treatise of Surveying, as they are not constructed in any Book on that Subject.

24. To construct Scales of Chords, Sines, Tangents, &c.

1st. With a convenient Extent of the Compasses, and one Point at C, describe a Semicircle, ADB, Pl. I. Fig. and raise the $\perp$ CD; then with the same Extent of the Compasses, and one Point in D, with the other make a Mark, H, on the Arc DGHB; then, with the same Extent, one Leg being in B, the other will extend to G, and the Arc DB be divided into 3 equal Parts; and $\therefore$ DG, GH, and HB, are each = 30 Degrees; then, by a Trial or two, divide each of these into three equal Parts, and so the Quadrant will be divided into 9 equal Parts, each 10 Degrees. 276. g.

2. Produce AB towards F at Pleasure, and draw DE $\parallel$ thereto; then, turning a Ruler about C as a Center, draw Lines to touch the Tangent Line DE, viz. a 10, b 20, G 30, &c. then it is manifest, from Art. 7, that DE will be a Scale of Tangents.

3. Then, one Leg of the Compasses being in C, open the other to 10 on the Tangent-Line, and, turning it about, sweep an Arc, 10, 10, to cut the Line CF; then sweep Arcs, 20, 20; 30, 30; &c. and CF becomes a Scale of Secants.

4. Parallel

Construction of Scales.

4. Parallel to DC draw the Parallels, *a* 10, *b* 20, &c. to cut the Line CB; then is CB a Scale of Sines; and, if the Numbers were counted from B towards C, a Scale of Versed Sines, by Art. 6.

5. Place one Leg of the Compasses in D, and, extending the other to *a*, describe the Arc *a* 10; in the same Manner, *b* 20, &c. to cut the Line DB; then will the Line DB be a Line of Chords, by Art. 5.

6. Lastly. If A, D, be joined by a Line, AD, and the Arc AD divided into 8 equal Parts, denoting Points of the Compass used in Navigation, and Arcs 1, 1; 2, 2; &c. be described in the same Manner as in the Line of Chords; the Line AD becomes a Scale of Chords for Points, commonly called, by Sailors, a Line of Rumbs.

25. Though, in the above, we have only divided the Arc DB into 9 Parts, denoting 10 Degrees each, (being sufficient for an Example,) it is easy to conceive that each of these may be divided into 10 Degrees, and so the Arc divided into 90 Degrees, and consequently the Scales made to Degrees, &c.

26. But, though the above Construction clearly shews the Nature of such Scales, yet, as it requires very great Care to be accurate, it is not so fit for Practice as the constructing them by Means of Tables, &c. as I shall shew farther on, being unwilling to detain the young Student any longer in this Place, and desirous of making him acquainted with the Solution of the common Cases of Plane Triangles, as soon as conveniently may be.

27. Having a Set of Scales, accurately made, called Pattern-Scales, readily to make others by it.

For Example: Let it be required to transfer the
Pl. II. Graduations on the Scale AB, to the intended Scale
Fig. 1. DC.

1st. Draw the necessary parallel Lines from D to C, by the sharp Point of a Gage, of the same Nature as those used by the Cabinet-makers or Joiners.

2. Fix

Construction of Scales.

9

2. Fix the two Scales by the Side of each other, on a Table or proper Board, so that they may not slide.

3. Then take a Square, *abe*, in which *ab* is a Ruler, or Piece of Mahogany or other proper Wood, in the Middle of which is laid a Brass Square, *cde*.

Its Use is: Slide *ab* along the Side of the Pattern-Scale AB, till the Edge *de* comes to a Division, suppose 2, to be transferred to the other Scale; which is readily done by only cutting with a sharp-pointed Knife, by the Edge *de* of the Square on the Scale DC; and so sliding from one Graduation to another, the Divisions on AB are easily copied on the intended Scale DC. As to the Figures, if on Wood, they are marked by proper Stamps; if on Brass, they should be engraved.

28. To make an $\angle =$ any Number of Degrees.

This has been already done by a Line of Chords, and may also be done by Scales of Sines, Tangents, or Secants.

For Example: Let it be required to make an $\angle$ equal to 50 Degrees.

1st. By the Line of Sines.

On any Point, C, in the Line AB, raise the $\perp$ CD = the Sine of 50d. taken from the Scale of Pl. II. F. Sines; then, from the same Scale, take the Radius 2. or Sine of 90d. and, having one Point of the Compasses in D as a Center, with the other intersect AB in E, by a little Sweep; join ED; and the $\angle$ DEB is manifestly that required.

2. By the Line of Tangents.

Take AC = the Radius or Tangent of 45d. from the Scale of Tangents; raise the $\perp$ CD, which Pl. II. F. make = the Tangent of 50 Degrees; draw AD;^{3.} and the $\angle$ DAB is that required.

3. By the Secants.

Take, as before, AC = Radius, (= Secant of Pl. II. F. 0 Degrees = Sine of 60 d. = Tangent of 45°,) and^{4.} raise $\perp$ CE; then with the Secant of 50 d. and one Leg of the Compasses in A, with the other sweep a

C
little

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little Arc, intersecting CE in D ; and it is evident the $\angle$ DAB is = the required $\angle$.

B O O K I. C H A P. III.

Of the Solution of the common Cases of Right-angled Plane Triangles.

Pl. II.
Fig. 5.

29. *Definition.* In any right-angled Triangle, ABC, right-angled at B, the Side AB, on which the Triangle may be supposed to stand, is called the Base ; and the Side BC, from its Position, the Perpendicular. The Side opposite the Right-Angle, or the longest Side, AC, is named the Hypothenuse.

30. By attentively considering Article 4, these Propositions evidently follow ; *viz.*

Pl. II.
Fig. 6.

1st. If the Hypothenuse be made Radius, then is the Perpendicular BC the Sine of the opposite Angle A ; and the Base AB the Cosine of the Angle A ; or, which is the same, the Sine of the Angle C.

Pl. II.
Fig. 7.

2dly. If the Base be made Radius, then is the Perpendicular the Tangent of the Angle A, and the Hypothenuse the Secant of the Angle A.

3dly. If the Perpendicular be made Radius, then is the Base the Tangent of the Angle C, and the Hypothenuse the Secant of the Angle C.

31. *Scholium.* We have noted the above, as we intend to shew how to solve right-angled Triangles by making each Side Radius, as some may expect it ; but we should hint, that, in Practice, Navigators, Surveyors, &c. generally make the Hypothenuse Radius, when it can be done (which it may always, excepting when the two short Sides or Legs, *viz.* the Base and the Perpendicular, are given to find the Hypothenuse) ; because it is more easily remembered, &c. as it comes under the following universal Theorem,* *viz.* In any Plane Triangle, if
among

* Which will be demonstrated in Oblique Triangles.

among the Things given there be a Side, and its opposite Angle, to find the Rest : then say, As a given Side : the Sine of its opposite Angle :: any other Side : the Sine of its opposite Angle.

Therefore, to find an Angle, begin with a Side opposite to a known Angle. Also, as the Sine of a given Angle : its opposite Side :: the Sine of any other Angle : its opposite Side.

Therefore, to find a Side, begin with an Angle opposite to a known Side.

32. *Corollary 1.* Hence, if the opposite Sides are equal, the opposite Angles are equal ; and the contrary.

33. *Corollary 2.* Hence also the greater Angle is opposite to the greater Side, and the contrary ; agreeably to Theorem the 6th, in the Introduction.

34. The six Cases of Right-angled Trigonometry may be easily solved, by observing the following Directions : *Viz.*

1. Either make a Draught of the Triangle by Geometry, or (which may be sufficient) only a Representation, drawn in a rough Manner by Hand.

2. Let such Parts of the Triangle as are given in the Question be marked with a Dash (—), and those which are to be found, with a Cypher (o).

3. If one of the Acute-Angles is given, the other is also looked on as known ; being found by only subtracting the given one from 90 d. This appears from Art. 83, *Elements of Geometry*.

4. Compare the given Things together, and thereby determine which of the preceding Rules is best adapted to solve the Problem.

5. Then write the Canon as the Theorem directs, and, if you use a Table of Natural Sines, Tangents, and Secants, work the Proportion as in the Rule of Three in common Arithmetic ; but, if a Table of the Logarithms of these Natural Sines, Tangents, &c. place the Canon in four Lines under one another ; and, taking out of the proper Ta-

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bles the Logarithms of the given Quantities, add the Logarithms of the second and third Terms together, and subtract the Logarithm of the first from the Sum, and the Remainder will be the Logarithm of the fourth Term.

For Logarithms are Numbers artificially contrived, in such a Manner, that their Addition answers to Multiplication, and Subtraction of them to Division of Natural Numbers.

35. *N. B.* The Index of the Logarithm of any Number is always one less than the Number of Integral-Places in the given Number; and, consequently, the Number of Integral-Places of Figures in any Number is always one more than the Index of its Logarithm.

If the young Student is inclined to see the Doctrine of Logarithms more fully treated of, he may consult our Essay on that Subject. The Tables of Logarithms are too voluminous to be given in this Volume, as they cannot conveniently be comprized in less than 135 Pages; therefore, if the young Student has not a good Set already by him, we must refer him to our *British Mariner's Assistant*, just published, (Price Six Shillings,) where, amongst 40 Tables adapted to the several Purposes of Trigonometry and Navigation, he will find very accurate Tables of Logarithms to *seven* Places of Decimals. However, as we may perhaps frequently have Occasion to solve some practical Problem in Trigonometry, when Tables of Logarithms are not at Hand, I have added, at the End of this Essay, a Table of Natural Sines, Tangents, and Secants, to 4 Figures; and, if we have Occasion to solve a common Question in the Absence of any Books, (which may sometimes be the Case,) I have contrived a Nautical Pocket-Piece, about the Size of a Crown, shewing the Natural Sines and Tangents to 3 Places, which is sufficiently exact for most Uses at Sea, &c.

to be had of the Publisher of this Essay, Price Sixpence.

36. The common Cases of right-angled Δ s are :

Case 1. Given the Angle at the Base, and the Hypotenuse, to find the Base and the Perpendicular.

EXAMPLE. Let the Angle at A be 40 d. 15', and the Hypotenuse 130 Feet : Quære the other two Sides ?

Pl. II.
Fig. 8.

Construction 1. Draw a Line, AB, at Pleasure.

2. Make an $\angle A =$ the given $\angle$ 40 d. 15', and draw AC, which make equal to the Hypotenuse, 130 Feet.

3. From C let fall the Perpendicular CB ; then it is manifest that the right-angled Triangle ABC is that required.

37. If AB and BC be measured on the same Scale of equal Parts as AC was set off from, you will have nearly the Lengths of the Base and Perpendicular ; but more accurately by Calculation, as follows.

C A N O N S.

38. 1st. Making the Hypotenuse Radius.

To find the Base.	To find the Perpendicular,
As Rad. or S. of 90° 10.0000000	As Radius 90° 10.0000000
Is to AC 130 Feet 2.1139433	Is to AC 130 Feet 2.1139433
So is S. $\angle C$ 49° 45' 9.8826568	So is S. $\angle A$ 40° 15' 9.8103159
To AB 99.22 Feet 1.9966001	To BC 84 Feet 1.9242592

By Natural Sines thus. By the Table, at the End of this Essay, the Sine of 49° 45' is .7632, and the Sine of 40° 15' is .6461 ; therefore say, by the Golden Rule, As Radius 1 : 130 :: .7632 : .7632 $\times$ 130 = 99.2, nearly, the required Base ; and as 1 : 130 :: .6461 $\times$ 130 = 84, nearly, the Perpendicular which was to be found.

39. Note, $\angle C$ 49 d. 45' was found by subtracting $\angle A$ 40 d. 15' from 90 d. If the Learner is inclined,

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clined, he may save himself the Trouble of subtracting, by taking out the Cosine of $40^{\circ} 15'$; because, the Angles A and C being always $= 90^{\circ}$, the Cosine of the $\angle$ A must be equal to the Sine of the Angle C.

40. In the above Canons, the first Logarithm is to be subtracted from the Sum of the two middle Logarithms; which, as the first Logarithm is 10, is here done by only cutting off 1, in the Place of Tens, in the Sum.

41. 2. Making the Base Radius.

As Sec. $\angle A 40^{\circ} 15'$ 10.1173432	As S. $\angle A 40^{\circ} 15'$ 10.1173432
Is to AC 130 Feet 2.1139433	Is to AC 130 Feet 2.1139433
So is Radius 90° 10.0000000	So Tan. $\angle A 40^{\circ} 15'$ 9.9276590
12.1139433	12.0416023
10.1173432	10.1173432
To AB 99.22 Feet 1.9966001	To BC 84 Feet 1.9242591

Or thus, by the Table of Natural Secants, &c.

By the Table, the Natural Secant of $40^{\circ} 15' = 1.309$
And the Natural Tangent of $40^{\circ} 15' = .8466$

Then, as $1.309 : 130 :: 1 : \frac{130 \times 1}{1.309} = 99.3$,
nearly, the Base required.

Again, as $1.309 : 130 :: 0.8466 : \frac{0.8466 \times 130}{1.309}$
 $= 84$, nearly, = the required Perpendicular.

42.

* If the Table of Logarithms we are using has not a Table of Secants, take out the Log. Cosine of $40^{\circ} 15'$, viz. 9.8826568, and, subtracting it from 20, we readily get 10.1173432, the required Log. Secant of $40^{\circ} 15'$. And in the same Manner the Secant for any other Number of Degrees may be found.

42. 3. Making the Perpendicular Radius.

As Sec. $\angle C 49^{\circ} 45'$ 10.1896841	As Sec. $\angle C 49^{\circ} 45'$ 10.1896841
Is to AC 130 Feet 2.1139433	Is to AC 130 Feet 2.1139433
So Tan. $\angle C 49^{\circ} 45'$ 10.0723410	So is Radius 90° 10.0000000
<u>12.1862843</u>	<u>12.1189433</u>
10.1896841	10.1896841
<u>To AB 99.22 Feet 1.9966002</u>	<u>To BC 84 Feet 1.9242592</u>

Or thus.

The Natural Secant of $49^{\circ} 45'$ being 1.547
And the Natural Tangent of $49^{\circ} 45'$ being 1.181

We have, As 1.547 : 130 :: 1.181 : $\frac{1.181 \times 130}{1.547}$
= 99.2, nearly, = the required Base.

And, lastly, As 1.547 : 130 :: 1 : $\frac{130 \times 1}{1.547}$ =
84, nearly, = the Perpendicular which was required.

43. In like Manner may the following Cases be solved by Natural Sines, Tangents, &c. as well as by Logarithms ; but as, in Order to be very nearly accurate, Tables must be given, calculated to every Minute of the Quadrant or 90 Degrees, it would take up more Room than we could spare ; and, as they must be computed to 6 or 7 Places of Decimals, the Operations of multiplying and dividing would become more tedious than by Logarithms : We shall, therefore, for the Future, give only the Solution by Logarithms, it being sufficient to have shewn the Method of Solution by Natural Numbers.

N. B. The Natural Sines are given to 7 Places of Figures, for every Minute of the Quadrant, in our *British Mariner's Assistant*.

44. Case

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44. *Case 2.* Given the Angle at the Base, and the Base, to find the Perpendicular and Hypothenufe.

Pl. II.
Fig. 9.

EXAMPLE. Let the Angle at A be 50 d. and the Base 86 Yards ; to find the Perpendicular and Hypothenufe.

Construction. Make $\angle A = 50$ d. $AB =$ Base, 86 Yards : On the Point B erect the Perpendicular BC, to intersect AC in C : Then is ABC the Triangle required.

Calculation. — 1. Making the Hypothenufe AC Radius.

45. To find the Perpendicular.	To find the Hypothenufe.
As S. $\angle C$ 40° 9.8080675	As S. $\angle C$ 40° 9.8080675
Is to AB 86 Yards 1.9344984	Is to AB 86 Yards 1.9344984
So is S. $\angle A$ 50° 9.8842540	So is Radius 90° 10.0000000
<u>11.8187524</u>	<u>11.9344984</u>
<u>9.8080675</u>	<u>9.8080675</u>
To BC 102.5 Yards 2.0106849	To AC 133.8 Yards 2.1264309

46. 2. Making the Perpendicular Radius.

As Tang. $\angle C$ 40° 9.9238135	As Tang. $\angle C$ 40° 9.9238135
Is to AB 86 Yards 1.9344984	Is to AB 86 Yards 1.9344984
So is Radius 90° 10.0000000	So is Sec. $\angle C$ 40° 10.1157460
<u>11.9344984</u>	<u>12.0502444</u>
<u>9.9238135</u>	<u>9.9238135</u>
To BC 102 5 Yards 2.0106849	To AC 133.8 Yards 2.1264309

47. 3. Making the Base Radius.

As Radius 90° 10.0000000	As Radius 90° 10.0000000
Is to AB 86 Yards 1.9344984	Is to AB 86 Yards 1.9344984
So is Tan. $\angle A$ 50° 10.0761865	So is Sec. $\angle A$ 50° 10.1919325
To BC 102.5 Yds 12.0106849	To AC 133.8 Yds 12.1264309

48. Case 3. Given the Perpendicular and the Angle at the Base, to find the Base and Hypothenufe.

EXAMPLE. Let the $\angle$ at A be $33^{\circ} 45'$, and the $\perp$ 75 Feet; required the Base and Hypothenufe.

Pl. II.
Fig. 10.

Construction. Make $\angle A = 33^{\circ} 45'$, and draw AD; then, $\parallel$ to AB, at a Distance = 75 Feet, draw EF, to intersect AD in C; from C let fall the $\perp$ CB: Then it is evident the $\triangle ABC$ is that required.

Otherwise, 1st, By subtracting the $\angle A$ $33^{\circ} 45'$ from 90° find the $\angle C = 56^{\circ} 15'$. 2. Assume a Point, B, in the Line AG, and raise the $\perp BC = 75$ Feet. 3. Make $\angle C = 56^{\circ} 15'$, and draw CA, to intersect HG in A; and the $\triangle$ is constructed.

Calculation. — 1. Making the Hypothenufe Radius.

49. To find the Base.	To find the Hypothenufe.
As S. $\angle A$ $33^{\circ} 45'$ 9.7447390	As S. $\angle A$ $33^{\circ} 45'$ 9.7447390
Is to BC 75 Feet 1.8750613	Is to BC 75 Feet 1.8750613
So is S. $\angle C$ $56^{\circ} 15'$ 9.9198464	So is Radius 90° 10.0000000
11.7949077	11 8750613
9.7447390	9.7447390
To AB 112.3 Feet 2.0501687	To AC 135 Feet 2.1303223

Of Right-angled Plane Triangles.

50. 2. Making the Perpendicular Radius.

As Radius 90° 10.0000000	As Radius 90° 10.0000000
Is to BC 75 Feet 1.8750613	Is to BC 75 Feet 1.8750613
So Tan. $\angle C$ $56^{\circ}15'$ 10.1751074	So Sec. $\angle C$ $56^{\circ}15'$ 10.2552610
To AB 112.3 Feet 12.0501687	To AC 135 Feet 12.1303223

51. 3. Making the Base Radius.

As Tan. $\angle A$ $33^{\circ}45'$ 9.8248926	As Tan. $\angle A$ $33^{\circ}45'$ 9.8248926
Is to BC 75 Feet 1.8750613	Is to BC 75 Feet 1.8750613
So is Radius 90° 10.0000000	So Sec. $\angle A$ $33^{\circ}45'$ 10.0801536
11.8750613	11.9552149
9.8248926	9.8248926
To AB 112.3 Feet 2.0501687	To AC 135 Feet 2.1303223

Pl. II. 52. Case 4. Given the Base and Hypothenufe, to
Fig. 11. find the Angles and Perpendicular.

EXAMPLE. Let the Base be 300 Feet, and the Hypothenufe 500 Feet: Quære the Angles and Perpendicular.

Construction. Make $AB = 300$ Feet, and raise $\perp$ BD ; then, with an Extent in the Compasses $= 500$ Feet, and one Point in A , with the other describe an Arc, to intersect BD in C ; join A, C , and $\triangle ABC$ is that required.

Calculation. — 1. Making the Hypothenufe Radius.

53. To find the Angles.	To find the Perpendicular.
As AC 500 Feet 2.6989700	As Radius 90° 10.0000000
Is to Radius 90° 10.0000000	Is to AC 500 Feet 2.6989700
So is AB 300 Feet 2.4771212	So is S. $\angle A$ $53^{\circ}7'48''$ 9.9030895
12.4771212	To BC 400 Feet 12.6020595
2.6989700	
To S. $\angle C$ $36^{\circ}52'12''$ 9.7781512	

54. 2. If the Perpendicular be Radius, there is no Angle given, related to a given Side; and therefore this Case cannot be solved by making the Perpendicular Radius.

55. 3. Making the Base Radius.

As AB 300 Feet	2.4771212	As Radius 90°	10.0000000
Is to Radius	10.0000000	Is to AB 300 Feet	2.4771212
So is AC 500 Feet	2.6989700	So T. ∠ A 53° 7' 48"	10.1249372
	12.6989700	To BC 400 Feet	12.6020584
	2.4771212		
To Sec. ∠ A	10.2218488		
53° 7' 48"			

56. *Scholium.* When the Perpendicular is required, and the Angles are not required, it may be found more expeditiously thus. Add the Logarithms of the Sum and Difference of the Hypothenuse and Base together; Half that Sum will be the Logarithm of the Perpendicular. Thus, in the above Example, the Sum of the Hypothenuse and Base = 800; their Difference = 200.

Logarithm of 800 = 2.9030900

Ditto of 200 = 2.3010300

5.2041200

$\frac{1}{2}$ 2.6020600 This answers in the Table to 400, the $\perp$ required.

The Reason of this is evident from the Nature of Logarithms, and Art. 305, *Elements of Geometry*.

57. *Case 5.* Given the Perpendicular and Hypothenuse, to find the Angles and Base.

EXAMPLE. Let the Perpendicular be 80 Yards, and the Hypothenuse 100 Yards: Quære the Angles and Base.

Construction. Draw a Line, DB, and at B raise the $\perp$ BC = 80 Yards; then, with the Extent of the D 2 Hypothenuse,

Pl. II.
Fig. 12.

Of Right-angled Plane Triangles.

Hypothenufe, 100 Yards, and one Leg in C, with the other describe a little Arc, to intersect DB in A; join A, C, and the Triangle ABC is that required.

Calculation. — 1. Making the Hypothenufe Radius.

58. To find the Angles.

As AC 100 Yards 2.0000000

Is to Radius 90° 10.0000000

So is BC 80 Yards 1.9030900

11.9030900

2.0000000

To S. ∠ A 53° 7' 48" 9.9030900

To find the Base.

As Radius 90° 10.0000000

Is to AC 100 Yards 2.0000000

So S. ∠ C 36° 52' 12" 9.7781523

To AB 60 Yards 1.7781523

59. 2. Making the Perpendicular Radius.

As BC 80 Yards 1.9030900

Is to Radius 90° 10.0000000

So is AC 100 Yards 2.0000000

12.0000000

1.9030900

To Sec. ∠ C }
36° 52' 12" } 10.0969100

As Radius 90° 10.0000000

Is to BC 80 Yards 1.9030900

So T. ∠ C 36° 52' 12" 9.8750628

To AB 60 Yards 1.7781528

60. 3. If the Base be made Radius, we have no Angle given, related to a given Side; and therefore this Case cannot be solved by making the Base Radius.

61. *Scholium.* To find the Base, independent of the Angles, add the Logarithms of the Sum and Difference of the Hypothenufe and $\perp$, and take Half the Sum for the Logarithm of the Base.

The

Of Right-angled Plane Triangles.

23

The Sum of the Hypothenuſe and Perpendicular
is 180 Yards; its Logarithm . 2.2552725
Their Difference = 20, its Log. 1.3010300

3.5563025

$\frac{1}{2}$ 1.7781512 This
answers, in the Table of Logarithms, to 60, the
Baſe required. The Reason of this is the ſame as
in the laſt Caſe.

62. *Caſe 6.* Given the Baſe and Perpendicular,
to find the Angles and Hypothenuſe.

EXAMPLE. Let the Baſe be 120 Feet, and Per-
pendicular 90 Feet: Quære the Hypothenuſe and
Angles.

Conſtruction. Make AB = the Baſe, 120 Feet;
and, on B, raiſe a $\perp$ BC, which make = 90 Feet;
join A, C, and the Δ ABC is evidently that re-
quired.

Pl. II.
Fig. 13.

Calculation. — 1. Making the Hypothenuſe Ra-
dius.

Here is no given Angle related to a given Side,
and therefore it cannot be ſolved by making the
Hypothenuſe Radius.

63. 2. Making the Perpendicular Radius.

To find the Angles.

As BC 90 Feet 1.9542425
Is to Radius 90° 10.0000000
So is AB 120 Feet 2.0791812
12.0791812
1.9542425
To T. $\angle C$ 53° 7' 48" 10.1249387

To find the Hypothenuſe.

As Radius 90° 10.0000000
Is to BC 90 Feet 1.9542425
So Sec. $\angle C$ } 10.2218477
53° 7' 48" }
To AC 150 Feet 12.1760902

Of Right-angled Plane Triangles.

64. 3. Making the Base Radius.

As AB 120 Feet	2.0791812	As Radius 90°	10.0000000
Is to Radius 90°	10.0000000	Is to AB 120 Feet	2.0791812
So is BC 90 Feet	1.9542425	So Sec. $\angle A$	10.0969105
	11.9542425	36° 52' 12"	
	2.0791812	To AC 150 Feet	12.1760917
ToT. $\angle A$ 36° 52' 12"	9.8750613		

65. *N. B.* We may find the Angle as above, and then the Hypothenufe may be found by this Canon. As S. $\angle A$: BC :: Radius : the Hypothenufe.

66. *Scholium.* The common Cafes of right-angled Triangles might have been reduced to a much smaller Number, but these will not be found too many for the Pupil's Instruction.

B O O K II.

Of Oblique Triangles.

67. *Definition.* Every Triangle which has not a Right-Angle is called an Oblique Triangle.

68. Before we proceed to the Cafes of Oblique Triangles it is proper to premise a few Theorems, by Way of Lemmata, as the Solutions of the several Cafes depend on them.

Theorem 1. In all Plane Triangles the Sides are proportional to the Sines of their opposite Angles ;* *et contra.*

Note,

Pl. II. * *Demonstration.* Take $AF = BC$, and let fall the $\perp$ s CD, Fig. 14. FE ; let BC be made Radius, then is also AF Radius : Then, by

Note, this is the same as is more largely expressed in Art. 20.

69. *Theorem 2.* If, from the Half Sum of any two Numbers, be taken their Half Difference, the Remainder will be the lesser Quantity; but the Half Sum, added to the Half Difference, gives the greater Number.†

70.

by right-angled Triangles, FE is the Sine of the Angle A, and CD the Sine of the Angle B. The Lines, FE and CD, being both $\perp$ to AB, are parallel to each other; and $\therefore$ the Triangles AEF, ADC, are similar. Hence, as AF : FE :: AC : CD. But AF = BC by the Construction, FE the Sine of the Angle A, and CD the Sine of the Angle B; $\therefore$ the above Analogy becomes, BC : S. $\angle$ A :: AC : S. $\angle$ B. Q. E. D.

Corollary 1. Hence it follows, that, if two Δ s have one $\angle$ in one = one $\angle$ in the other, and the Sides opposite to those equal Angles are equal, that will have the greatest Base whose opposite Angle is nearest to a Right-Angle.

For, by this Theorem, as S. $\angle$ A : BC :: S. $\angle$ C : AB. Hence, as A and BC are constant, the greater the S. $\angle$ C is, the greater will AB be: but the S. $\angle$ C is greatest when the $\angle$ C is a Right-Angle; $\therefore$ AB will be greatest when the $\angle$ C is a Right-Angle.

Corollary 2. Hence also, if the Angles A and C are equal, their opposite Sides must be equal; and the contrary.

† *Demonstration.* Let $2x$ represent the greater, and $2y$ the lesser, Number. Then their Sum is $2x + 2y$, or their Half Sum $x + y$; and their Difference $2x - 2y$, or their Half Difference $x - y$.

To $x + y$
Add $x - y$

From $x + y$
Sub. $x - y$

$2x$. Q. E. D. Remains $+ 2y$. Q. E. D.

That x added to x is $2x$, and that y being to be added to x , and subtracted from x , will destroy each other, and so that $2x$ is the Sum of $x + y$ and $x - y$, is sufficiently clear; but, perhaps, to such as have not learnt Algebra, it may not be so evident that $x - y$, subtracted from $x + y$, should leave $2y$. If so, let it be considered that we are only to find the Difference betwixt $x + y$ and $x - y$; now $x + y$ denotes a Quantity, which is y , greater than x ; and $x - y$ a Quantity, y , less than x ; consequently, as one is y greater, and the other y less, their Difference must be $2y$.

Geometrically

70. *Corollary*. From hence it follows, that, if from the greater of any two Quantities be taken their Half Sum, the Remainder will be equal to their Half Difference.

71. *Theorem 3*. In any Plane Triangle, as the Sum of any two Sides is to their Difference, so is the Tangent of Half the Sum of the two opposite Angles to the Tangent of Half their Difference. ‡

72. *Theorem 4*. As the lesser Side is to the greater, so is Radius to the Tangent of an Angle : From that Angle subtract 45° , and note the Remainder. Then, as Tangent of 45° (or Radius) is to the Tangent of that Remainder, so is the Tangent of Half

Pl. II.
Fig. 15. *Geometrically thus*. Let the greater Quantity be represented by AC, the lesser by CB; then is AB their Sum; which bisect in D; so is AD or DB equal to Half their Sum. Take AE = CB; then, as AC is one Quantity, and AE the other, EC is manifestly the Difference, and $\therefore$ ED = DC, the Half Difference of the Quantities. Hence it plainly appears, that the Half Sum AD, + the Half Difference DC, is equal to the greater Quantity AC; and that the Half Sum, — DC the Half Difference, is = BC, the lesser Quantity. Q. E. D.

Pl. II.
Fig. 16. † *Demonstration*. Let ABC be the Triangle. About C, as a Center, with the Radius CB, describe the Circle EBDE; produce AC to D; and join EB and BD. About E, as a Center, with the Radius EB, describe the Arc Bd, and with the same Radius, about B, as a Center, the Ark Ea; and draw EF $\perp$ to BE. Then is AD = AC + CB, the Sum of the two Sides, and AE = AC — CB, (or CE,) the Difference of the two Sides by the Construction. The $\angle$ BAC + $\angle$ ABC = $\angle$ BCD*. But $\angle$ BED = $\frac{1}{2}$ $\angle$ BCD, by Art. 145 of our *Geometry*; $\therefore$ $\angle$ BED = $\frac{1}{2}$ Sum of the Angles A and ABC. Now, the $\angle$ EBD, being in a Semicircle, is a Right-Angle, by Art. 148 of our *Geometry*; and $\therefore$ BD, the Tangent of the $\angle$ CEB, by Art. 7.

* 76. g. Again: $\angle$ CBA — $\angle$ CBE (or CEB) = $\angle$ EBA, the $\frac{1}{2}$ Difference of the Angles ABC and A, by Art. 70; of which EF is the Tangent. But, by similar Triangles, AD the Sum of the Sides : AE the Difference of the Sides :: BD the Tangent of Half the Sum of the Angles at the Base : EF the Tangent of Half their Difference. Q. E. D.

Half the Sum of the two opposite Angles to the Tangent of Half their Difference.*

73. This Theorem serves for the same Purpose as the third, only when we have the Logarithms of the Sides given, instead of the Natural Numbers, as is the Case in calculating the Places of the Planets from astronomical Tables; it is Something readier in Practice.

74. *Theorem 5.* In any Plane Triangle, as the Base is to the Sum of the other two Sides, so is their Difference to the Difference of the Segments of the Base.†

E 75.

* *Demonstration.* Let ACb be a right-angled Triangle, right-angled at C , and having two Sides, AC , Cb , respectively = the two Sides, AC , CB , of the Triangle ABC . Then, by Theorem 3d, in the right-angled Triangle ACb , As $AC +$

$$Cb : AC - Cb :: (T, \frac{\angle AbC + \angle CAb}{2} = \text{because the } \angle$$

$$ACb \text{ is a Right-Angle}) T, 45^\circ : T, \frac{\angle AbC - \angle CAb}{2}$$

$$(\text{but } 90^\circ - \angle AbC = \angle CAb, \therefore \text{ substituting for } CAb, \text{ we get, } T, \frac{\angle AbC - \angle CAb}{2} = T, \frac{2 \angle AbC - 90^\circ}{2}) = T,$$

$$\frac{\angle AbC - 45^\circ}{2}. \text{ Also, by Theorem 3d, in the Triangle } ABC,$$

$$\text{As } AC + CB : AC - CB :: T, \frac{\angle ABC + \angle CAB}{2} : T,$$

$$\frac{\angle ABC - CAB}{2}; \therefore \text{ by Equality, as } T, 45^\circ : T, \frac{\angle AbC - 45^\circ}{2}$$

$$:: T, \frac{\angle ABC + \angle CAB}{2} : T, \frac{\angle ABC - \angle CAB}{2}; \text{ which}$$

is the second Part of the Theorem. Q. E. D.

Lastly, by the 6th Case of Trigonometry, As bC (CB) : $AC ::$ Radius : Tangent of $\angle AbC$; which is the 1st Part of the Theorem. Q. E. D.

† *Demonstration.* Let ABC be the Triangle, AB the Base. About C , as a Center, with the Radius CB , describe the Circle. Pl. II. Fig. 17.

75. *Corollary.* This Theorem gives also twice the Distance of the Perpendicular from the Middle of the Base.

For, if we put m to represent the Middle of the Base, and b Half the Base; then $AE = b + mE$, and $BE = b - mE$; and, consequently, the Difference of the Segments $= 2mE$.

76. The common Cases of oblique Triangles are,
Case 1st. Given two Angles and one Side, to find the other Sides. This is solved by *Theorem 1*.

Pl. III. Fig. 1. *EXAMPLE.* Let the Base AB be 150 Yards, the $\angle A$ 30 d. and $\angle B$ 40 d. Quære AC and BC .

Construction. Make $AB = 150$ Yards, $\angle A = 30$ d. $\angle B = 40$ d. and draw AC and BC to intersect in C ; then is the Triangle ABC that required.

Calculation. The $\angle A$ 30 d. + $\angle B$ 40 d. = 70 d. and 180 d. - 70 d. = 110 d. = $\angle C$.

To find BC.		To find AC.	
As S. $\angle C$ 110°	9.9729858	As Sup. $\angle C$ 70°	9.9729858
Is to AB 150 Yards	2.1760913	Is to AB 150 Yards	2.1760913
So is S. $\angle A$ 30°	9.6989700	So is S. $\angle B$ 40°	9.8080675
	11.8750613		11.9841588
	9.9729858		9.9729858
To BC 79.81 Yards	1.9020755	To AC 102.6 Yards	2.0111730

77.

cle $GFBDG$; join C, F ; biseft FB by the Line CE ; then, CB being $= CF$, $FE = EB$, by Construction, and CE common, the Triangles FEC and BEC are equal in every Respect, by our *Geometry*, Art. 95. Therefore $\angle FEC = \angle CEB$; or, in other Words, $CE \perp$ to AB . The two Parts, into which the Base is cut by the $\perp$ CE , are called *Segments*.

In our Essay on Geometry, Art. 212, it is demonstrated, that $AB \times AF = AD \times AG$. Hence this Analogy, As AB ; $AD :: AG : AF$. But AB is the Base; and it is plain, from the Construction, that AD is = the Sum of the Sides AC and CB ; $AG =$ their Difference, *viz.* $= AC - CG$ (or CB); and $AF = AE - EF$ (or EB) = the Difference of the Segments.

The above Analogy, expressed in Words, will be as in the Theorem.

77. *Case 2.* Given two Sides, and an Angle opposite to one of them, to find the other Side and Angles.

EXAMPLE. Let $AB = 82$ Feet, $AC = 66$ Feet, the $\angle B = 49^{\circ} 30'$; quære the Side BC , and Angles A and C . Pl. III.
Fig. 2.

Construction. Make $AB = 82$ Feet; $\angle B = 49^{\circ} 30'$: then, taking a Distance of 66 Feet between the Compasses, place one Leg on A , and turn the other about to intersect the Line BD , which it will do in two Places, C and E ; so that this Case admits of two Solutions; for either of the Triangles, ABC or ABE , has the Conditions required in the Question, and therefore this is called the doubtful Case, unless it is known, from some Circumstance, whether the Angle opposite AB is obtuse or acute. If acute, Theorem 1st finds the Angle E ; if obtuse, its Supplement to 180° is the Angle C : for it has been already observed, that any Arc and its Supplement to 180 Degrees have the same Sine.

In the following Solution we will suppose the Angle opposite to AB to be obtuse, and so ABC to be our Triangle.

CANONS.

To find the $\angle$ s C and A .

As AC 66 Feet 1.8195439

Is to $S. \angle B$ $49^{\circ} 30'$ 9.8810455

So is AB 82 Feet 1.9138139

To $Sup. \angle C$ $70^{\circ} 52'$ 11.7948594

Hence $\angle C = 180^{\circ} - 70^{\circ} 52' = 109^{\circ} 8'$; and $\angle A = 180^{\circ} - 109^{\circ} 8' - 49^{\circ} 30' = 21^{\circ} 22'$.

To find BC .

As $S. \angle B$ $49^{\circ} 30'$ 9.8810455

Is to AC 66 Feet 1.8195439

So is $S. \angle A$ $21^{\circ} 22'$ 9.5615010

11.3810449

9.8810455

To CB 31.62 Feet 1.4999994

78. *Case 3.* Given two Sides and the Angle contained between them, to find the other Angles and Side.

Of Oblique Triangles.

The Angles are found by Theorems 3d and 2d ; then the other Side by Theorem 1st.

Pl. III. Fig. 3. EXAMPLE. Let the Angle $A = 22^{\circ} 30'$, $AC = 300$ Yards, and $AB = 400$ Yards : Quære the Angles at B and C, and Side BC.

Construction. Make $AB = 400$ Yards, $\angle A = 22^{\circ} 30'$, $AC = 300$ Yards ; join B, C ; and the Triangle ABC is that required.

To and from AB 400	From 180 0
Add and subtract AC 300	Subtract $\angle A \quad 22 \quad 30$
Gives Sum of Sides 700	Gives $\angle B + \angle C \quad 157 \quad 30$
Their Difference 100	Half Sum of the $\angle s \quad 78 \quad 45$

C A N O N S.

To find the Angles.

As Sum of Sides } 700 Yards .	2.8450980
Is to their Dif. 100 Y. 2.0000000	
So is Tan. $\frac{1}{2}$ Sum unknown $\angle s$ }	10.7013382
$78^{\circ} 45'$.	
	12.7013382
	2.8450980

To Tan. of $\frac{1}{2}$ their Diff. $35^{\circ} 41'$ }	9.8562402

To and from $78^{\circ} 45'$	
Add and subtract $35 \quad 41$	
Gives $\angle C \quad 114 \quad 26$	
$\angle B \quad 43 \quad 4$	

To find BC.

As S. $\angle B \quad 43^{\circ} 4'$	9.8343246
Is to AC 300 Yards	2.4771213
So is S. $\angle A \quad 22^{\circ} 30'$	9.5828397
	12.0599610
	9.8343246
To BC 168.12 Yds	2.2256364

This Case may also be solved by Theorem 4th, and by letting fall a Perpendicular, as in Pl. III.

Fig.

Fig. 4, independently of either of these Theorems ; for in the right-angled Triangle ADC will be given the Hypotenuse AC, and Angle A, to find, by Right-Angle Trigonometry, the Base AD, and Perpendicular DC. Then $AB - AD = BD$; and therefore, in the right-angled Triangle BDC, are given BD and DC, to find the Hypotenuse BC.

79. Case 4. Given the three Sides, to find the Angles.

Method of Calculation. From the greater Angle let fall the Perpendicular DC ; with the Radius CD describe the Arc BE ; then are BD and AD called Segments ; and, BD being = DE, AE is the Difference of the Segments, which is found by Theorem 4th : Then from AB subtract AE, and the Remainder is = EB ; Half of which is ED, or its Equal DB, the short Segment.

Pl. III.
Fig. 4.

Now, the Triangle is divided into two right-angled Triangles ; in each of which are given the Hypotenuse and Base, to find the Angles, by Case 4th of right-angled Triangles, or by Theorem 1st. Lastly, the $\angle ACD$, added to the $\angle DCB$, will give the whole $\angle ACB$; or the $\angle$ s A and B, added together, and subtracted from 180, are = the $\angle ACB$.

EXAMPLE. Given $AB = 350$ Feet, $AC = 240$ Feet, $BC = 200$ Feet ; to find the Angles.

The Triangle is constructed by the Method shewn in Art. 224, of the *Essay on Geometry*.

The Operation by the above Directions will be as under.

$$AC + BC = 440, \text{ and } AC - BC = 40.$$

CANONS.

Of Oblique Triangles.

C A N O N S.

To find the Difference of the Segments.

As AB 350 Feet 2.4540680	From AB = 350
Isto AC + BC 440 F. 2.6434527	Subtract AE = 50.29
So is AC - BC 40 F. 1.6020600	EB = 299.71
4.2455127	ED or DB $\frac{1}{2}$ the short } 149.85
2.5440680	Segment . . . }
	Add AE = 50.29
To AE Diff. } 1.7014447	Gives AD long Segment 200.14
of Segments, }	
50.29 Feet . }	

To find the Angles A
and ACD.

As AC 240 Feet 2.3802112
Is to Radius 90° 10.0000000
So is AD 200.14 Feet 2.3013339
12.3013339
2.3802112
To S. $\angle$ ACD } 9.9211227
56° 30' . }
And 90° - 56° 30' ($\angle$ ACD)
= 33° 30' = the $\angle$ A.

To find the Angles B
and BCD.

As CB 200 Feet 2.3010300
Is to Radius 90° 10.0000000
So is DB 149.85 Feet 2.1756567
12.1756567
2.3010300
To S. $\angle$ DCB } 9.8746267
48° 32' . . }
And 90° - 48° 32' ($\angle$ DCB)
= 41° 28' = the $\angle$ B.

To find the Angle C.

The $\angle$ A 33° 30' + $\angle$ B 41° 28' = 74° 58';
and 180° - 74° 58' = 105° 2' = $\angle$ C.

Or, $\angle$ ACD 56° 30' + $\angle$ DCB 48° 32' = 105° 2'
= $\angle$ C.

B O O K III.

Of the improved Navigation-Scale.

80. **H**AVING, in many Years Teaching, observed, that the *Navigation-Scale*, commonly called, by Sailors, *Gunter's Scale*, was generally inaccurately made, and very deficient in Lines of equal Parts, I long wished for an Improvement thereon; and, in the Year 1764, being in *London*, I took that Opportunity to have some more accurately made, with some Improvements of my own; and, in the Beginning of the Year 1772, being again in Town, I readily embraced that Opportunity to make some farther Improvements, and a new Pattern, with the Assistance of a good Workman; so that I flatter myself, if compared with the common Scales, they will be found more generally useful and accurate. Several incorrect pirated Copies, made in a very unworkmanlike Manner, and some with even the Diagonals drawn wrong, having been sold in *London* and *Bristol*, and perhaps in some other Places, as my improved Scales; to prevent as much as possible my own Character from being injured thereby, and the Public from being imposed on, I hereby desire that any of the principal Shops, inclined to sell them, will apply to me to know with whom I have left the Pattern for the Trade, and to put, under *Improved by B. Donn*, their own Names as *Makers*; and the Public may be assured, that, whoever shall offer to Sale the Scale without my Pamphlet of its Use, is selling a pirated Copy of the Scale, and therefore not to be depended on.

81. The several Particulars, in which this Scale, as now improved, excels the common Scales, will plainly appear, from the following Description and Use,

Of the improved Navigation-Scale.

Use, to those who will take the Trouble to compare it with the common Scales.

82. The Plane or Plotting Side, *viz.* that which has the Bevil Edge, contains the following Scales.

1. A Scale of Inches, divided into *Tenths* of an Inch. On the Left, or at the Beginning of the Inches, are several Brass Pins, with Figures annexed, which shew the Weight of the Ball to any Diameter of the Gun. For Example: If the Diameter of the Bore of a Gun be 5 Inches and 4 Tenths, it shews by Inspection that such a Bore will carry 18 Pound Shot. This Addition is made, as it may be useful to Numbers of Sailors; for whose Use it may be proper to hint, that the Weight of Powder for Service is generally about Half the Weight of the Shot.

2. Under the Line of Inches, a Foot is divided into 100 equal Parts, so as by Inspection to turn readily Inches into the Decimal of a Foot, or the contrary. Its Use is known to such as are acquainted with Mensuration. Dimensions may also be taken in Feet and decimal Parts, instead of Feet and Inches, &c. and so the Content of any Piece of Timber, &c. found by common Multiplication. For Example: Suppose a Plank is 20 Feet long, and 1 Foot 50 Hundredths broad; then 1.50, multiplied by 20, gives 30 Feet, the Content required. *This Line is not on the common Scales.*

3. The common Scales contain but few Scales of equal Parts; by which Means the young Navigator and Surveyor was frequently at a Loss for a proper Scale to construct his Scheme with. To remedy this, I have seen some Scales made, by the Direction of an ingenious Gentleman, which contained more Scales of equal Parts than the common ones; but, in Order to make Room for them, the Diagonal Scales were omitted. But, in Cases which require particular Accuracy in constructing, the Diagonal Scales are necessary; therefore

fore I have contrived to give even two more Scales, and yet retained the Diagonal-Scales, by disposing of the Lines in different Order. The Scales of equal Parts are 10, 15, 20, 25, 30, 35, 40, 45, 50, and 60, to an Inch.

4. This Side contains likewise Lines of Rumba, Chords, Sines, Tangents, &c. as on the common Scales; also two Lines, corresponding to each other, one marked *M. Lon.* the other *Chord*, by which are shewn, by Inspection, how many Miles make a Degree of Longitude in any Latitude. For Example, against 60 Degrees of Longitude, on the *Chord*, you will find, on the *M. Lon.* that 30 Miles make a Degree of Longitude in that Latitude.

I shall now proceed to describe the Contents of the other Side, viz. that which is commonly called *Gunter's*, from the Logarithms of Numbers, Sines, and Tangents, being first laid thereon by Mr. Edward Gunter.

83. For the Sake of the young Student, it may be proper to observe, that, to work a Canon or Proportion on *Gunter's* Scale; we have only to extend the Compasses from the first Term to the second, on the proper Line, then will the same Extent, laid the same Way from the third Term, (viz. from the Left to the Right, or from the Right to the Left, according as the second Term lay from the first,) give, on its proper Line, the fourth or required Number.

84. As, in the common Rule of Three, it is no Matter which of the two middle Terms is placed first, since the Product is the same, so, in working on the *Gunter's* Scale; if at any Time it is found more convenient, we may extend the Compasses from the first Term of the Proportion to the third Term, then will that Distance extend from the second Term to the fourth or required Term.

85. In counting on the Line of Numbers, it may be proper to observe, that the Numbers 1, 2, 3, 4, &c. represent not only 1, 2, 3, 4, &c. but 10, 20,

Of the improved Navigation-Scale.

30, &c. Thus, if the Canon you are working of requires you to call the 1, in the Middle of the Line of Numbers, 100, then must the 2, 3, 4, 5, 6, 7, 8, 9, 10, on the Right-Hand, be called 200, 300, 400, 500, 600, 700, 800, 900, 1000; the 9, 8, 7, 6, 5, 4, 3, 2, 1, on the Left, be called 90, 80, 70, 60, 50, 40, 30, 20, 10.

If the 1 in the Middle be called 1000, the Figures 2, 3, 4, &c. on the Right, will be 2000, 3000, 4000, &c. and 9, 8, 7, 6, &c. on the Left, 900, 800, 700, 600, &c.

Again, if the 1, in the Middle of the Line of Numbers, be called 10, the Figures 2, 3, 4, &c. on the Right-Hand, must be 20, 30, 40, &c. and, on the Left, 9, 8, 7, &c. will be only 9, 8, 7, &c.

Also, if the 1 in the Middle be called 1, then 2, 3, 4, &c. on the Right-Hand, will stand for 2, 3, 4, &c. but 9, 8, 7, &c. on the Left, will be only $\frac{9}{10}$, $\frac{8}{10}$, $\frac{7}{10}$, &c. &c.

86. For the young Student, the Canons, in common Use in working a Day's Work, are stamped on the Scale. Thus $R : \text{Diff.} :: SC : \text{Dep.} :: SCC : D. \text{ Lat.}$ signifies, that, as Radius is to the Distance, so is Sine of the Course to the Departure, and so is Sine-Complement of the Course to the Difference of Latitude.

Again, $D. \text{ Lat.} : \text{Dep.} :: \text{Tan. } 45 : T. \text{ Course,}$ is, as Difference of Latitude is to the Departure, so is Tangent of 45 Degrees to the Tangent of the Course.

Also, $SC \text{ mid. Lat.} : \text{Dep.} :: R : D. \text{ Long.}$ is to be read, as Sine-Complement of middle Latitude is to the Departure, so is Radius 90 Degrees to the Difference of Longitude. Or, to find the Difference of Longitude by meridional Parts, $D. \text{ Lat.} : \text{Dep.} :: M.D. \text{ Lat.} : D. \text{ Long.}$ which is to be read, as Difference of Latitude is to the Departure, so is meridional Difference of Latitude to the Difference of Longitude.

Lastly,

Lastly, $SCL : S \odot Dec. :: R : S \odot Amp.$ is to be read, as Sine-Complement of Latitude is to the Sine of the Sun's Declination, so is Radius, or Sine of 90 Degrees, to the Sine of the Sun's Amplitude.

87. The *first Line* on the *Gunter's Side* is marked *S. Rumb*, that is, Sines of Rumbs or Courses.

88. The *second Line* is marked *Numb. Sqr.* and is the common Line of Numbers intended to be used with the Sines or Tangents, &c. in working the usual Canons in Trigonometry or Navigation. It is marked *Numb. Sqr.* because, with the Line marked *Root*, it will serve for squaring Numbers, &c. of which farther Notice will be taken presently.

89. The *third Line*, marked *S | CS | Sec.* is the common Line of Sines, with the Addition of being numbered backward by small Figures; by which Addition it now serves the several Purposes of a Line of *Sines*, *Co-sines*, and *Secants*. A Notion has pretty generally prevailed, amongst Teachers of Trigonometry and Navigation, that the Canons, in which Secants are concerned, could not be solved by *Gunter's Scale*; and also amongst some Writers on Navigation; for Mr. K——y, in his *Navigator's Tutor*, and Mr. M——y, in his *Rudiments of Navigation*, both positively affirm, that, *If there is a Secant in the Proportion, the Operation cannot be performed by the Scale.* But that they are mistaken may be readily shewn, by only working one Example. Let us suppose the Angle A, or Angle at the Base of a Rt. $\triangle$ to be given, equal to 50 Degrees, and the Base 86 Yards, to find the Hypothenufe. If the Base be made Radius, the Canon is, as Radius is to the Base 86, so is Secant-Angle, A 50 Degrees, to the Hypothenufe.

This may be worked on the Scale thus: Extend the Compasses from Radius, or Sign of 90 Degrees, to 50 Degrees, on the same Line, counting backward from Radius, by the small Figures, for the Secant; then, one Leg of the Compasses being set at

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86, on the Line of Numbers, that Extent will, turned the contrary Way, (because the Secants must be conceived to run beyond the Scale,) reach to 134, the Distance which was required.

90. The *fourth* and *fifth* Lines are the versed Sines and Tangents, as on common Scales.

91. The *sixth* is the Meridian-Line, which, on the common Scale, is too short to be of much Service, but, by making it on two Lines, we are enabled to go to 80 Degrees of Latitude, and yet have the Degrees sufficiently large to divide the Meridian of WRIGHT'S Chart into every 10 Minutes, or less Parts. — The Length of the Degree of the Equator, corresponding to these of the Meridian, is annexed on the Left-Hand, and marked *Eq. Deg.*

92. The Meridian-Lines being constructed by the diagonal Scale of 20, if, at any Time, in the Absence of Tables, we are inclined to know nearly the Meridional-Difference of Latitude between any two Places, we have only to take the Distance between the two Latitudes from the Meridian-Line, then, measuring the Distance on the diagonal Scale of 20, we shall have nearly the meridional Miles required. Thus, for Example, if it were required to find the meridional Diff. of Lat. between the Latitudes of 20 and 30 Degrees, the Distance between these Degrees, taken from the Meridian-Line, will, on the lesser diagonal Scale, measure 663 (reckoning each primary Division, or Half-Inch, 100 Miles) for the Meridional-Difference of Latitude.

93. The *seventh* and *eighth*, or two remaining Lines of this Scale, are a single and triple Line of Numbers, marked *Numb. Root*, *Numb. Cube*, which, together with the second Line, or double Line of Numbers, marked *Numb. Sqr.* are for *squaring* or *cubing* Numbers, or for extracting the *Square* and *Cube Root*, or for working *Proportions wherein Squares and Cubes, or the Square and Cube Roots, are concerned*, of
great

great Use in various Parts of the Mathematics. We shall give a few Examples of their Use.

94. EXAMPLE 1. What is the Square-Root of 144? or, which is the same Thing, what is the Side of a Square whose Area is 144 Feet?

Solution. Call the 1, at the Beginning of the Line of Numbers, (marked *Numb. Sqr.*) 100, and extend the Compasses from that Point to 144; then, calling the 1, at the Beginning of the Line of Numbers, (marked *Numb. Root.*) 10, and putting one Point of the Compasses in that Point, the other will extend to 12, the required Root.

95. EXAMPLE 2. *The Areas of Circles being as the Squares of their Circumferences*, let us suppose the Weight of Cables, of unequal Circumferences but of equal Lengths, to be in the same Proportion. On this Supposition, let it be required to find the Weight of a Cable whose Circumference is 8 Inches; the Weight of one, of equal Length, of 10 Inches Circumference, being 25 Hundred. Here, by the Supposition, the Proportion is, as the Square of 10 is to the Square of 8, so is 25 to the required Weight.

To solve this on the Scale, extend the Compasses, on the Line *Numb. Root.* from 10 to 8, then, on the Line *Numb. Sqr.* will that Extent reach from 25 to 16 Hundred, the Weight of the Cable of 8 Inches.

96. EXAMPLE 3. What is the Cube-Root of 1728? or, which is the same Thing, if a Cube or Die contains 1728 solid Inches, what is the Side of the Cube?

This may be solved by a bare Inspection; for, calling the 1, at the Beginning of the Line *Numb. Cube*, 1000, and the 1 corresponding to it, on the Line *Numb. Root*, 10, then, against 1728, on *Numb. Cube*, you will see 12, the required Root, on *Numb. Root*.

97. EXAMPLE 4. Suppose a Ship of 300 Tons to be 75 Feet by the Keel, it is required to find the Length

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Length of the Keel of a similar Ship of 500 Tons Burthen.

Solution. *Similar Solids being in Proportion as the Cubes of their like Sides*, we have, as 300 is to 500, so is the Cube of 75 to the Cube of the required Keel. Therefore, extend the Compasses, on the Line *Numb. Cube*, from 300 to 500, (or from 3 to 5,) then will that Extent, on the Line *Numb. Root*, reach from 75 to 89 Feet nearly, the Length of the Keel required. This Instance of a useful but troublesome Question, if performed by Arithmetic, being so expeditiously solved on the Scale, by the Addition of these Lines, is a sufficient Recommendation of their Use.

98. What we have said on the Use of the improved Scale is sufficient for those who are acquainted with the Use of the common Scales; as for others, it would be adviseable for them to apply to a proper Master for Instructions. However, I intend to treat more fully of the Uses of the several Lines, with their Constructions, in my Treatise of Navigation, to which the Subject more naturally belongs.

99. To prevent the Public being imposed on, by the Sale of pirated Scales, it may be proper to acquaint them that I have appointed the following Mathematical-Instrument-Makers, in London, to vend them, *viz.* Messieurs *Adams, Heath and Wing, Lincoln, Martin, Nairne, Watkins, and Whitford*: As they will not sell any without the Pamphlet of their Use, and their own Names, as Makers, stamped on the Scales, consequently, their own Reputation will not permit them to sell any made in an inaccurate or unworkmanlike Manner.

BOOK

BOOK IV. *Of Altimetry and Longimetry.*CHAP. I. *Of Altimetry.*

100. **A**LTIMETRY is the Application of the Doctrine of Triangles to find the Heights of Objects, whether accessible or not.

101. If an Object stands on a Plane, so that a Line may be measured from the Observer to the Foot of the Object, it is said to be accessible, otherwise not.

102. *Prop. 1.* To find the Height of an accessible Object.

This may be done several Ways, as in the following Cases.

Case 1. To find the Height, when the Sun (or Moon) shines.

EXAMPLE. Let it be required to find the Height of a Tower, the Length of whose Shadow, on the horizontal Plane, was 90 Feet, at a Time when the Shadow of an upright Pole, of 10 Feet Height, was 9 Feet.

PL. III.
Fig. 5.

Construction. Let AB represent the Plane; make $AD = 9$ Feet, and $\perp DE = 10$ Feet: Also $AB = 90$ Feet; and raise the $\perp BF$; produce AE to intersect it in C; then it is plain BC must be the Height of the required Tower.

Solution. It is manifest, the Δ s ADE, ABC, are similar: $\therefore$ as AD, the Shadow of the Pole, : DE, the Height of the Pole, :: AB, the Shadow of the Tower, : BC, the Height of the Tower. In this Example, as $9 : 10 :: 90 : 100$, the Height of the required Tower.

103. *Case 2.* To find the Height, by a Walking-Stick or Pole, when the Sun does not shine.

EXAMPLE. Suppose I want to find the Height of the Tree CE, above the horizontal Plane AC, by Means of a Pole of 15 Feet in Length.

Stick

Pl. III.
Fig. 6.

Stick the Pole $BF \perp$ to the Plane AC , at a convenient Place; then walk backwards from it to A , till you see the Points F and E in a right Line: This done, measure from A to B , also from A to C . Let us suppose $AB = 10$ Feet, $AC = 20$ Feet, and the Height of the Observer's Eye, $AH = 5$ Feet.

Construction. Make $AB = 10$ Feet and the $\perp BF = 15$, also $AC = 20$ Feet, and erect $\perp CI$, which intersect by HF produced in E ; then is CE evidently = the Height of the Tree required.

Solution. Draw $HD \parallel$ to AC ; then it is manifest the Δ s HGF and HDE are similar. Therefore, as $HG (AB) : GF (BF - BG) :: HD (AC) : DE$. In our Example, as $10 : 10 :: 20 : 20 = DE$. Then $DE + DC (AH) = 20 + 5 = 25$ Feet, the Height of the Tree required.

104. *Case 3.* To find the Height by Means of a Bowl of Water, or other horizontal reflecting Surface.

Pl. III.
Fig. 7. *Method.* Let A represent the Plane of the reflecting Surface, DB the horizontal Plane: Walk backwards to D , till your Eye, at E , sees the Reflection of the Top of the Object BC in the Basin A ; then measure DA , which suppose $= 7$ Feet, and $AB = 70$ Feet, the Height of the Eye, $DE = 5$ Feet.

Construction. Make $AD = 7$ Feet, $AB = 70$ Feet, and erect $\perp$ s. Make $DE = 5$ Feet, and $\angle BAC = \angle DAE$, then will BC be the Height of the Object required; because, by the Nature of Reflection, the $\angle$ s DAE and BAC must be always = each other. See Art. 317 in our Essay on Geometry.

Solution. The $\angle$ s ADE , ABC , being manifestly = each other, we have, as $AD : DE :: AB : BC$, that is, in the above Example, as $7 : 5 :: 70 : 50 = BC$, the Height of the Object required.

105. *Case 4.* To find the Height, by Means of a Quadrant or other proper Instrument to take the Angle of Elevation, of the Top of the Object.

EXAMPLE. Suppose, standing at A , the Eye at D , the Angle of Elevation EDC is taken $= 40$ Degrees; and

and from A to B the Foot of the Object measures 100 Feet: Quære the Height of the Object BC, the Eye D being 5 Feet above the horizontal Plane AB.

Construction. Make $AB = 100$ Feet, the $\perp AD = 5$ Feet; also erect the $\perp BF$, and draw DC, making an Angle EDC, $= 40^\circ$. Then is BC evidently = the Height of the Object required. Pl. III.
Fig. 8.

Solution. By Trigonometry, as Cosine $\angle D$, or Sine $\angle C$, : $DE = AB :: S \angle D : EC$: Then $EC + EB (DA) = BC$, the required Height.

Operation by Logarithms.

As Cosine 40° .	9.8842540	
Is to 100 Feet	2.0000000	
So is Sine 40° .	9.8080675	$EC = 83.91$ Feet
	11.8080675	
	9.8842540	$EB = 5$ Feet
To $EC 83.91$ Feet	1.9238135	$BC = 88.91$ Feet

106. *Scholium.* As, perhaps, some of our Readers may not be acquainted with the Nature of a Quadrant, it may be proper to describe that commonly made Use of in Altimetry.

Construction. On a Piece of Brass, or dry Board that will not easily warp, make a Quadrant, or Quarter of a Circle, ABC. Draw $ab \parallel$ to AB. About a , as a Center, describe the Arc bc ; and, since the Radius is = the Chord of 60° Degrees, with the same Extent, and one Point of the Compasses in b , with the other, make a Mark in the Arc at e ; then bisect the Arc be in d ; take the Distance bd and set from e to c and join ac ; then is $bd = de = ec = 30^\circ$ Degrees; and therefore the whole Arc $bdec = 90^\circ$ Degrees, or a Quarter of a Circle. See the F.
in Pl. IV.

G

By

By a Trial or two, divide the Arc bd , de , and ee , each into 3 equal Parts, or every 10 Degrees; then bisect each of these into every 5 Degrees; lastly, by a few Trials, into every Degree; which, if the Quadrant be sufficiently large, as a Foot Radius, may be subdivided into Halves, Quarters, or lesser Divisions, viz. into every Minute, by Diagonals, as will be explained when we describe the Quadrants used at Sea. Make $\angle bas = \angle far = 45^\circ$. draw $fb \parallel ag$ and $gf \parallel ab$, then will $agfb$ be a Square, by the Construction. Divide gf , fb , each into 100 or 1000 equal Parts. From the Center a hang a Plumb-Line aD ; the finer the Line is the better, a $\dagger$ Horse-Hair will do very well; the Weight may be either Lead or Brass; and, if the Line be hung by a little Bow, on a Pin at a , instead of being run through a Hole, it will play better.

It only remains to fix two convenient Sights, ik , on the Edge of the Quadrant, AC , or on any Line $\parallel$ thereto. These are commonly made only of two Bits of Brass, with a little Hole in each, of the same Height: But as, by these Means, it is very difficult to see the Object through the farthest Sight, by Means of the small Aperture or Opening of the Hole, I generally make them in the Manner represented by the Figures i and k in the Plate. The large Apertures are made sufficiently open to take in a large Field of View, and have Cross-Hairs or fine Silver Wires intersecting each other at right $\angle$ s, so that each Intersection is exactly of the same Height as the Hole in the Sight next the Eye.

If the Quadrant be too large to hold steady by Hand, it may be fixed to a Ball and Socket, and supported by the Legs of a common Surveying-Table, (generally called a Plain-Table,) or to a Pole like that of a Barber's Block.

107. The Manner of using this Quadrant, to take an Angle of Altitude of any Object, is thus:

Holding

$\dagger$ Or fine Silver Wire.

Holding the Quadrant upright in your Hands, look through the Sights to the Top of the Object C, then will the Number of Degrees, cut by the Plumb-Line, viz. the Arc GH, express the Angle of Altitude.

Pl. III.
Fig. 9.

Demonstration. Draw $DE \parallel AB$. Then, DE being $\parallel$ to the Horizon, by the Nature of a Plumb, the $\angle DIF$ must be a Right $\angle$; and so the $\triangle DIF$ a right-angled one; therefore, (by 82 of Geometry) the $\angle EDF$ (IDF) + $\angle HFD$ (IFD) = a Right $\angle$; and $\angle GFD$ being also a Right-Angle, the $\angle GFH$ + $\angle HFD$ = a Right $\angle$. Therefore, taking away the $\angle HFD$, common to both, the $\angle EDF$: (the Angle of Elevation) is = $\angle GFH$ = the Number of Degrees in the Arc GH. $\angle E D.$

The Angle of Depression of an Object, L , viz. the $\angle LFK$, (FK being $\parallel BA$), is found in nearly the same Manner, by only placing the Eye at the Center of the Quadrant, F , (instead of at D), and looking through the Sights FD to L . Then is the $\angle GFH$ = the $\angle$ of Depression, LFK .

108. Case 5. To find the Height of an accessible Object by Means of the Quadrant, which is of great Use when absent from Tables or proper Instruments for calculating.

The Method is this. Look at the Top of the Object through the Sights of the Quadrant, as before explained; then, instead of looking the Degrees on the Arc, count the Divisions between r and d cut by the Plumb-Line; which reserve: Conceive a Line, DE , $\parallel$ to AB , and $BC \perp$ to both: Then, the $\angle$ s DEC , deb , being Right $\angle$ s, and $\angle edb$ = $\angle$ of Elevation, EDC , (as has been already shewn, in the Use of the Quadrant,) it follows, edb must be = $\angle DCE$, by Art. 83 Geometry. Hence the $\triangle$ s edc , DEC , are similar: $\therefore As bc :: ed :: DE :: EC$. This is when the $\angle$ of Elevation is less than 45° d. or the Plumb falls between r and e , or on the Side of the Quadrant called Sinistra (or Left-Hand). But,

Pl. III.
Fig. 10.

Pl. III.
Fig. 11.

when the $\angle$ of Elevation is more than 45 d. or the Plumb-Line falls between a and e , on the Side called Dextra, or Right-Hand, then, by similar Triangles, as $ad : ab :: DE : EC$.

Pl. III.
Fig. 10.

109. EXAMPLE. Standing at A, my Eye being at D, 5 Feet above the horizontal Plane AB, I found, by the Quadrant, that bc being divided into 1000 Parts, cd was = 500; and from A to the Middle of the Base of the Obelisk measured 80 Feet. Quære the Height of the Obelisk.

By the above, as $cb : cd :: DE (AB) : EC$.
in this Example, by the Golden-Rule, As 1000 : 500 : 80 : 40 Feet.

To EC = 40 Feet,

Add BE = 5 = AD, the Height of the Eye.

Gives 45 Feet, the Height required.

110. Prop. 2. To find the Height of an inaccessible Object, CE, standing erect on an horizontal Plane.

Case 1. By the Quadrant.

Pl. III.
Fig. 12.

Supposing a River, Ditch, or some other Obstruction, between the Foot of the Object and the Observer, so that the Distance BC cannot be measured on the Ground: Therefore, at B he can only take the $\angle$ DGE 50 d. and, walking backwards, measure a convenient Distance, BA, (suppose 67 Feet,) and then take the $\angle$ of Elevation, DFE = 35 d. Hence it is required to find the Height of the Object CE.

Construction. Make a Line FG = 67 Feet, and produce it towards D at Pleasure; make $\angle$ DGE = 50 d. $\angle$ DFE = 35 d. then will the Point of Intersection be the Top of the Object; and, letting fall the $\perp$ DE, produce it to C, so that DC be made = the Height of the Eye, which we suppose 5 Feet; then it is manifest CE is the Height required.

Calculation.

Calculation. First, $180^{\circ} - \angle DGE = \angle FGE$, in our Example, $= 130^{\circ}$ and $\angle DFE + \angle FGE = 165^{\circ}$ which, subtracted from 180° gives $\angle FEG = 15^{\circ}$. Hence, by Trig. as $S. \angle FEG 15^{\circ} : FG 67 \text{ Feet} :: S. \angle GFE 35^{\circ} : GE = 148.5 \text{ Feet}$. Now, in the Right $\triangle GDE$, we have, as Radius $90^{\circ} : GE = 148.5 \text{ Feet} :: S. \angle DGE = 50^{\circ} : DE = 113.7 \text{ Feet}$; to which adding $DC = 5 \text{ Feet}$, we have $CE =$ the Height of the Object required, $= 118.7 \text{ Feet}$.

If the young Student is inclined, he may find the Side EF by saying, as $S. \angle FEG : FG :: S. \angle FGE : FE$. Then, as Radius : $FE :: S. \angle DFE : DE$. Which coming out the same as before will be a Proof of the whole Work.

If BC be required, As Radius : $GE :: \text{Cosine } \angle DGE : GD = BC$.

III. *Case 2.* But, if no Tables are at Hand, then the Height may be found by the *Quadrat*.

Thus, suppose the Plumb cuts, on the Dextra-Side, 845 at B , and at A , on the Sinistra-Side, 700, counting each Side of the *Quadrat* 1000, and $AB 67 \text{ Feet}$. Quare the Heights.

*Rule.** From the Square of the Number, into which each Side of the *Quadrat* is divided, (that is, from 10000, if the Side is divided into 100 Parts, or from 1000000, if into 1000 Parts,) subtract the Product of the Numbers cut by the Plumb, on both

the

Demonstration. Let $a =$ each Side of the *Quadrat, $s =$ the Number cut by the Plumb-Line on the Sinistra-Side, $d =$ the Number on the Dextra-Side, $m = FG$, $x = GD$; then $FD = FG + GD = m + x$. Now, by Proposition 1, *Case 5*, As $a : s$*

$:: m + x : \frac{sm + sx}{a} = DE$: Also, As $d : a :: x : \frac{ax}{d} = DE$:

Therefore, $\frac{ax}{d} = \frac{sm + sx}{a}$. Hence, $a^2x = smd + sdx$. And $\therefore$

$x = \frac{smd}{a^2 - sd}$. Q. E. D.

the Dextra and Siniftra Sides, and reserve the Remainder. Lastly, divide the continued Product of the Numbers cut on the Dextra-Side, the Siniftra-Side, and Length FG, by the reserved Number; the Quotient will be = to GD: Then, by Proposition 1, Case 5, find DE.

In the above Example, we have, by this Rule, the numerical Operation thus:

Multiply	845	From	1000000	
By	700	Subtract	591500	
	<u> </u>		<u> </u>	
	591500		408500	
	<u> </u>		<u> </u>	
Multiply	591500	4085)396305(97 nearly, =GD.		
By	67			
	<u> </u>		28655	
	4140500		<u> </u>	
	3549000		60	
	<u> </u>			
	39630500			
	<u> </u>			

Now, by Prop. 1, Case 5, As 845 : 1000 :: 97 Feet : 114.8 Feet, nearly, = DE, to which adding the Height of the Eye, 5 Feet, we have the whole Height CE = 119.8 Feet.

112. Case 3. If the Numbers, cut by the Plumb-Line, are both on the Siniftra-Side, then the Rule is, multiply * the least Number cut on the Quadrant by FG, and divide the Product by the Difference of the two

**Demonstration.* Let a = each Side of the Quadrant, l = the lesser, g = the greater Number cut on the Siniftra-Side of the Quadrant, m = FG, x = GD; then $FD = m + x$. By Prop. 1,

Case 5, As $a : l :: m + x : \frac{lm + lx}{a} = DE$: Again, $a : g ::$

$x : \frac{gx}{a} = DE$. $\therefore \frac{gx}{a} = \frac{lm + lx}{a}$. Hence $gx = lm + lx$. Therefore, $x = \frac{lm}{g - l}$. Q. E. D.

two Numbers found on the Quadrat; the Quotient will be equal to GD. Then find DE by Proposition 1, Case 5.

EXAMPLE. Let the Numbers, cut on the Siniftra-Side of the Quadrat, be 49 and 74, counting the Side of the Quadrat 100, and $FG = 78$ Feet.

Solution. Proceeding by the above Rule,

Multiply	78	From	74	25)3822(152.88
By	49	Subtract	49	
	702		25	132
	312			72
	3822			220
				200

Now, by Prop. 1, Case 5, As 100 : 74 :: 152.88 Feet : 113.1 Feet = DE; to which adding the Height of the Eye, it gives CE. Or we may say, As 100 : 49 :: 78 + 152.88 : DE, as above.

113. Case 4. But if the Numbers, cut by the Plumb-Line, are both on the Dextra-Side, the Rule* is the same as in the last Case.

EXAMPLE. Let $FG = 54$, the Numbers cut on the Dextra-Side of the Quadrat 82 and 31, counting the Side 100.

Numerical

* *Demonstration.* Let $a =$ each Side of the Quadrat, $l =$ the lesser $g =$ the greater Number cut on the Dextra-Side of the Quadrat, $m = FG$, $x = GD$; then $m + x = FD$. By Prop. 1,

Case 5, As $l : a :: x : \frac{ax}{l} = DE$; and, As $g : a :: m + x : \frac{am + ax}{g} = DE$. Hence $\frac{x}{l} = \frac{m + x}{g}$. $\therefore x = \frac{lm}{g - l} =$

GD. Q. E. D.

Altimetry.

Numerical Operation.

Multiply	54	From	82	51)1674(32.82=
By	31	Subtract	31	GD.
	54	Divisor	51	144
	162			420
Dividend	1674			120
				181

By Case 5, As 31 : 100 :: 32.82 Feet : 106 Feet, nearly, = DE. Or, As 82 : 100 :: 54 + 32.82 : DE, as before.

114. *Proposition 3.* To find the Height of an accessible Object, AD, when the Ground rises from it.

Pl. III.
Fig. 13.

Method. Let F be the Place of the Eye of a Person standing at any convenient Place, G, on the Hill; then, supposing FC an horizontal Line, by a Quadrant, take the $\angle$ of Altitude of the Top at D, which will be equal to $\angle$ CFD, which, for Example-Sake, we will suppose = 20 d. and at the same Place take the $\angle$ of Depression = 40 d. = BFC, (of a Mark B, set up from A on the Side of the Object AD, = the Height of the Eye FG;) then measure from G to A, suppose = 100 Feet: This done, the *Construction* is, Draw a Line AH, and erect a $\perp$ AE. Take AB = Height of the Eye, suppose 5 Feet; make $\angle$ FBE = 90 d. — 40 d. and BF = 100 Feet: Lastly, make $\angle$ BFD = 40 d. + 20 d. = 60 d. then is AD the Height of the Object required.

Calculation. First 90 d. — $\angle$ DFC 20 d. = $\angle$ FDC 70 d. Hence, by Trigonometry, As S. $\angle$ FDB 70 d. : BF 100 Feet :: S. $\angle$ BFD = 60 d. : BD = 92.16 Feet; to which adding AB, the Height of the Eye, 5 Feet, gives AD, the Height of the Object required, = 97.16 Feet.

115. *Proposition 4.* To find the Height of an accessible Object on the Top of a rising Ground, as BD.

Suppose, wanting to find the Height of the Object BD, standing at A, the Eye at E, 5 Feet high, I take the $\angle$ of Elevation FED = 50 d. and (BC being set off = AE = 5 Feet) the $\angle$ of Elevation FEC = 10 d. then, measuring from A to B, I found AB = EC = 60 Feet.

Pl. III.
Fig. 14.

Construction. Make $\angle$ FEC = 10 d. and EC = 60 Feet: Through C let fall the $\perp$ CF, and produce FC upwards at Pleasure; make $\angle$ FED = 50 d. then is BD manifestly the Height of the Object required.

Calculation. First, 90 d. — $\angle$ FED 50 d. = 40 d. = $\angle$ D; and $\angle$ FED 50 d. — $\angle$ FEC 10 d. = 40 d. = $\angle$ CED. By Trigonometry, As S. $\angle$ D 40 d. : EC 60 Feet :: S. $\angle$ CED 40 d. : CD = 60 Feet; to which adding the Height of the Eye BC 5 Feet, it gives the Object's Height BD = 65 Feet.

116. *Proposition 5.* To find the Height of an Object on the Top of a Hill which is inaccessible.

Suppose, wanting to find the Height of an Object, DE, on a Hill, CD, which is of too steep an Ascent to be conveniently mounted, I therefore stood at a convenient Place on the horizontal Plane, viz. the Eye being at B, I took Angles of Altitude CBE = 61 d. CBD = 39 d. then, measuring backwards, took the Distance BA = 50 Feet; and, the Eye being at A, the Angles of Altitude CAE = 42 d. and CAD = 22 d. Hence it is required to find the Height of the Object DE.

Pl. III.
Fig. 15.

N. B. The Height of the Eye is of no Consequence in this Proposition.

Construction. Draw a horizontal Line ABC, and make $\angle$ CAD = 22 d. CAE = 42 d. set off AB = 50 Feet, and make the $\angle$ CBE = 61 d. $\angle$ CBD = 39 d. join the Points of intersection DE, then is DE the Height of the Object required.

H

Solution.

Solution. The $\angle ABE = 180 \text{ d.} - \angle CBE 61 \text{ d.} = 119 \text{ d.}$ and $\angle BAE 42 \text{ d.} + \angle ABE 119 \text{ d.} = 161 \text{ d.}$ therefore $\angle AEB = 180 \text{ d.} - 161 \text{ d.} = 19 \text{ d.}$

Or shorter thus: Producing ED to C, the $\triangle AEC$ is right- $\angle$ d by Supposition; and therefore the $\angle$ s AEC and BEC are Complements of the $\angle$ s CAE, CBE; $\therefore$ the $\angle AEB$ being the Difference of these Complements must be also = Difference of these $\angle$ s; $\therefore \angle CBE 61 \text{ d.} - \angle CAE 42 \text{ d.} = \angle AEB = 19 \text{ d.}$ as before.

Hence, in $\triangle ABE$, As S. $\angle AEB 19 \text{ d.} : AB 50 \text{ Feet} :: S. \angle BAE 42 \text{ d.} : BE = 102.8 \text{ Feet.}$

Now the $\angle CBE 61 \text{ d.} - \angle CBD 39 \text{ d.} = \angle DBE = 22 \text{ d.}$ and, the $\triangle BCD$ being right-angled at C, the $\angle BDE = 90 \text{ d.} - \angle CBD 39 \text{ d.} = 51 \text{ d.}$ and $\angle BDE = 180 \text{ d.} - \angle BDC 51 \text{ d.} = 129 \text{ d.}$ Hence, As S. $\angle BDE 129 \text{ d.}$ or Sine of its Supplement $51 \text{ d.} : BE 102.8 \text{ Feet} :: S. \angle DBE 22 \text{ d.} : DE = 49.5 \text{ Feet,}$ which was to be found.

117. *Proposition 6.* To find the Height of an inaccessible Object, when we can only measure on the Side of a Hill on which it stands.

Pl. V.
Fig. 1.

EXAMPLE. Wanting to find the Height of an inaccessible Object, CD, at a convenient Place, B, I set up a Pole or Mark, BE, equal to the Height of my Eye, AF, suppose 5 Feet. Conceiving FG and EH $\parallel$ to the Horizon, standing first at A, the Eye at F, I took the $\angle$ of Altitude GFD = $46^\circ 30'$ and GFE = 33 d. then measuring from A to B (or F to E) I found that Distance = 38 Feet; then, with the Eye at E, I took the $\angle HED = 61^\circ 30'$, and the $\angle$ to the Foot of the Object, viz. HEC, = 18 d. Hence it is required to find the Height CD.

Construction. Draw a Line, FG, to represent a horizontal one; make $\angle GFE = 33 \text{ d.}$ $GFD = 46 \text{ d. } 30'$; set off FE = 38 Feet; draw EH $\parallel$ FG; then make $\angle HEC = 18 \text{ d.}$ and $HED = 61 \text{ d. } 30'$; from the Point of Intersection, D, of the Lines FD, ED, let

let fall the $\perp$ DC, and it is evidently the Height of the required Object above the Hill.

Calculation. The $\angle$ GFD $46^{\circ} 30'$ — $\angle$ GFE 33° d. $= 13^{\circ} 30' = \angle$ EFD; and, for the Reason shewn in the last Proposition, the $\angle$ HED $61^{\circ} 30'$ — $\angle$ GFD $46^{\circ} 30' = \angle$ FDE $= 15^{\circ}$ d. Hence, in $\triangle$ FDE, we have, As S. $\angle$ FDE 15° d. : FE 38 Feet :: S. $\angle$ EFD $13^{\circ} 30'$: ED = 34.27 Feet. Now, $\angle$ HED $61^{\circ} 30'$ — $\angle$ HEC 18° d. $= \angle$ CED $= 43^{\circ} 30'$; and the $\triangle$ EHC being right-angled at H, the $\angle$ ECH $= 90^{\circ}$ d. — $\angle$ HEC 18° d. $= 72^{\circ}$ d. But $\angle$ ECH is the Supplement of $\angle$ ECD.

Hence, by Trig. As S. $\angle$ ECD, or, which is equal, the Sine of its Supplement, ECH 72° d. : ED :: S. $\angle$ CED $43^{\circ} 30'$: CD 24.88 Feet. Q. E. I.

118. *Proposition 7.* To find the Height of an inaccessible Object at the Foot of a Hill, when we can only measure down the Hill towards it.

EXAMPLE. Wanting to find the Height of the Object CF, the Eye being at A, I took the $\angle$ of Altitude EAF $= 18^{\circ} \frac{1}{2}$ d. and the $\angle$ of Depression EAB $= 21^{\circ} \frac{1}{2}$ d. then, measuring towards the Foot of the Object C, the Distance AB = 38 Feet; my Eye being then at B, I took the $\angle$ of Altitude DBF $= 48^{\circ}$ d. and of Depression DBC $= 31^{\circ}$ d. Quære the Height of the Object FC.

Pl. V.
Fig. 2.

Construction. Draw a horizontal Line AE, make $\angle$ EAF $= 18^{\circ} \frac{1}{2}$ d. and $\angle$ EAB $= 21^{\circ} \frac{1}{2}$ d. then set off AB = 38 Feet: Through B draw BD $\parallel$ AE, make $\angle$ DBF $= 48^{\circ}$ d. and $\angle$ DBC $= 31^{\circ}$ d. From the Point of Intersection, F, let fall the $\perp$ FE, which produce to C, to intersect the Line BC, then is CF the Height of the required Object.

Calculation. The $\angle$ EAF $18^{\circ} \frac{1}{2}$ d. + $\angle$ EAB $21^{\circ} \frac{1}{2}$ d. $= \angle$ FAB $= 40^{\circ}$ d. and, for the Reason shewn in Prop. 5, the $\angle$ DBF 48° d. — $\angle$ EAF $18^{\circ} \frac{1}{2}$ d. $= \angle$ AFB $= 29^{\circ} \frac{1}{2}$ d. Now, by Trigonometry, As S. $\angle$ AFB $29^{\circ} \frac{1}{2}$ d. : AB 38 Feet :: S. $\angle$ FAB 40° d. : BF = 49.61 Feet.

The $\angle FBD 48^\circ + \angle DBC 31^\circ = \angle FBC 79^\circ$
 and $90^\circ - \angle DBC 31^\circ = \angle BCD = 59^\circ$. Hence,
 by Trigonometry, As S. $\angle BCD 59^\circ : BF 49.61$
 Feet $::$ S. $\angle CBF 79^\circ : FC = 56.8$ Feet. *Q. E. I.*

119. *Proposition 8.* To find the Height of an inaccessible Object, CD, standing on a horizontal Plane, BC, when I can only measure on the Side of an opposite Hill, BA.

Pl. V. Fig. 3. *EXAMPLE.* Standing at A, I took the $\angle$ s of Depression $EAD = 7^\circ$. $EAC = 19^\circ$. $EAB = 31^\circ$, and, by measuring, found $AB = 87$ Feet. Quære the Height of the Object CD, and BC the Distance from the Foot of the Hill.

Construction. Draw a horizontal Line AE, make $\angle EAD = 7^\circ$. $\angle EAC = 19^\circ$. and $\angle EAB = 31^\circ$. set off $AB = 87$ Feet, and draw $BC \parallel AE$; at the Intersection C raise $CD \perp$ to BC; then is CD the Height required.

Calculation. The $\angle BCA = \angle EAC = 19^\circ$. and $\angle FBA$ being $= EAB = 31^\circ$. and the $\angle FBA$ being the Supplement of $\angle ABC$, we have, by Trigonometry, As S. $\angle BCA 19^\circ : AB 87$ Feet $::$ S. of $\angle ABC$ or Sine of the supplemental $\angle FBA 31^\circ : AC = 137.6$ Feet.

The $\angle EAC 19^\circ - \angle EAD 7^\circ = \angle DAC = 12^\circ$. and $90^\circ - \angle EAD 7^\circ = \angle ADE = 83^\circ$. Hence, As S. $\angle ADC$ or Sine of its Supplement $83^\circ : AC 137.6$ Feet $::$ S. $\angle DAC 12^\circ : DC = 28.83$ Feet. *Q. E. I.*

If the Distance BC be required, then $\angle EAB 31^\circ - \angle EAC 19^\circ = \angle BAC = 12^\circ$. Hence, As S. $\angle BCA 19^\circ : AB 87$ Feet $::$ S. $\angle BAC 12^\circ : BC 55.56$ Feet. *Q. E. I.*

120. *Proposition 9.* To find the Height of an Object opposite a House or Tower, by Angles taken at the Windows of the House, or at Top and Bottom of the Tower.

EXAMPLE. From a Window at G, my Eye being 5 Feet above the horizontal Plane AB, I took the $\angle$ of

01 Altitude $\text{CGE} = 41^{\circ} 10'$; then, going up to another Window, F, 15 Feet higher, I took the $\angle$ of Altitude $\text{DFE} = 27^{\circ} \frac{1}{2}'$. Quære the Height of the Object BE.

Construction. Draw a horizontal Line AB, and erect the $\perp$ AF = 15 + 5 = 20 Feet; make AG = 5 Feet; draw FD and GC $\parallel$ to AB; make the $\angle$ $\text{CGE} = 41^{\circ} 10'$, and $\angle$ $\text{DFE} = 27^{\circ} 30'$; then is the Point of Intersection E the Top of the Object: $\therefore$ let fall the $\perp$ EB, and it is manifestly the Height required. Pl. V.
Fig. 4.

Calculation. The $\angle$ $\text{CGE} 41^{\circ} 10' - \angle$ $\text{DFE} 27^{\circ} 30' = 13^{\circ} 40' = \angle$ FEG ; and $\angle$ $\text{EFD} 27^{\circ} 30' + 90^{\circ} = 117^{\circ} 30' = \angle$ EFG . Hence we have, As $\text{S. } \angle$ $\text{FEG } 13^{\circ} 40'$: GF 15 Feet :: $\text{S. } \angle$ $\text{GFE } 117^{\circ} 30'$: GE 56.32 Feet. And, As Sine of 90° : GE 56.32 Feet :: $\text{S. } \angle$ $\text{CGE } 41^{\circ} 10'$: CE 37.07 Feet. Hence $\text{BE} = 37.07 + 5 = 42.07$ Feet; the Height which was required.

If the Distance AB (GC) be required, As Radius 90° : GE 56.32 Feet :: $\text{Cosine } \angle$ $\text{CGE } 41^{\circ} 10'$ (or $\text{S. } \angle$ $\text{GEC } 48^{\circ} 50'$) : GC = AB = 42.39 Feet.

121. *Proposition 10.* To find the Height of an Object by the Barometer.

We think we cannot oblige our Readers more than by giving them the following Paper (from the *Philos. Trans.* Vol 64, Part 1, 1774) as a Solution to this Proposition, viz.

M. De Luc's Rule, for measuring Heights by the Barometer, reduced to the English Measure of Length, and adapted to Fahrenheit's Thermometer and other Scales of Heat, and reduced to a more convenient Expression. By the Astronomer-Royal.

M. De Luc, F. R. S. in a large and valuable Treatise upon the Barometer and Thermometer, lately published at Geneva, the Result of many Years Labour and Study, has given a Rule for the Measurement of Heights by the Barometer, deduced from

from his Experiments, and far more accurate than any published before ; since it appears that he could determine Heights by it generally to 10 or 15 Feet, and that the Error seldom, if ever, amounted to double that Quantity. This valuable Degree of Exactness he has obtained principally by detecting the Faults of the common Barometer, and, in Consequence, improving the Construction of it, and by introducing the Use of the mercurial Thermometer, to accompany that of the Barometer. The principal Faults, which he found in the common Barometers, arose from the Repulsion of the Quicksilver by the Glass Tube, from Air and Moisture admitted into the Tube, and from the Variations of the Density of Quicksilver by Heat and Cold ; another very considerable Error arose, in calculating Heights from the Barometer, by not allowing for the Changes of the Density of the Air, whose Gravity affords us this Measure of Heights, owing to Heat and Cold. The first Cause of Error (that of the Repulsion of the Tubes) he remedied by substituting a Syphon-Barometer instead of the simple upright Tube, the Repulsion of the two Legs of the Syphon counteracting itself : The Error, arising from Air and Moisture in the Tube, he cured by boiling the Quicksilver after it was put into the Tube, and other Precautions : The Errors, in the Estimation of the Heights, arising from the Changes of the Density of the Quicksilver and Density of the Air, by Heat and Cold, he shews how to correct by Allowances, depending on two Thermometers, one attached to the Frame of the Barometer itself, and the other made to be exposed to the open Air, to shew its Degree of Heat ; which Thermometers are to be noted both at the Top and Bottom of the Hill. Lastly, by a great Number of Experiments, made with accurate Barometers and Thermometers of his own Construction, he has deduced a Rule for calculating the Heights of Places, the Exactness of which he has sufficiently proved by

a large Table of Experiments. But this Rule is expressed in *French Measure*, and is adapted either to a Thermometer, whose freezing Point is 0, and that of boiling Water 80, or to Thermometers of particular Scales. It may be therefore useful to reduce *M. De Luc's Rule* to English Measure, and to adapt it to the Thermometer of *Fahrenheit's Scale*, which is generally used in this Country.

122. * *M. De Luc*, in the Winter-Season, heated the Air of his Room to as great a Degree as he could, and noted the Rise of the Barometer, owing to the Diminution of its Density, or specific Gravity, by Heat; he also noted the Height of the Thermometer, both before and after the Room was heated. Hence he deduced a Rule, that, when the Barometer is at 27 *French Inches*, which was the Case in this Experiment, an Increase of Heat, from freezing to that of boiling Water, will raise the Barometer 6 Lines, or $\frac{1}{34}$ th Part of the Whole. It is easy to see, that, when the Barometer is higher than 27 *French Inches*, this Variation will increase in the same Proportion, or will be always $\frac{1}{34}$ th of the Height of the Barometer; therefore, if the Height of the Barometer be called B, the Rise of the Barometer, for an Increase of Heat, from freezing to boiling Water, will be $\frac{B}{54}$; and, as it will be less for a less Difference of Heat,

therefore, if the Number of Degrees, marked on the Thermometer, between freezing and boiling Water, be called K, and the Rise of the Thermometer, from any given Point, be called

H, the correspondent Rise of the Barometer will be $\frac{B}{54} \times \frac{H}{K}$, by

the Increase of Heat from the given Point, by the Number of Degrees H. If the Heat, instead of increasing, was to decrease, then H would signify so many Degrees Decrease of Heat, and

the Barometer would sink by $\frac{B}{54} \times \frac{H}{K}$. The fixed Tempera-

ture of Heat, to which *M. De Luc* thought best to reduce his Observations of the Barometer, is $\frac{1}{4}$ th of the Interval, from freezing to boiling Water, above the former Point; and, if the Thermometer was higher than this Degree, he subtracted

B

* For the Sake of such of our Readers as do not understand algebraic Expressions, we have printed the greater Part of this Essay of Mr. MASKELYNE's on a less Type, that they may not be deterred from reading the practical Part, which is printed on a larger Type, and may be easily understood.

$\frac{B}{K} \times \frac{H}{K}$; if it was lower, he added it to the observed Height of the Barometer; and thus he obtained the exact Height of the Barometer, such as it would have been if the Density of its Quicksilver had been the same as answers to the fixed Degree of Temperature. He thus corrected the Height of both his Barometers, (that at the Bottom and that at the Top of the Hill,) for the particular Degree of Heat, indicated by a Thermometer attached to the Barometer, at each Station; for it might and would commonly happen, that the Degree of Heat would be different at the two Stations. The Heights of the Barometers, thus corrected, were what he made Use of in his subsequent Calculations. Calling these two Altitudes of the Barometer B and b, putting Log. B and Log. b for the Logarithms of B and b, taking only the four first Places of Figures, after the Characteristic, or considering the remaining Figures as Decimals, and putting C for the mean Height of a Thermometer, exposed to the Air at Top and Bottom of the Hill, the freezing Point being 0, and the Point of boiling Water at 80, he finds, by his Experiments, that the Height of the Hill will be given in French Toises, when C is 16 $\frac{1}{2}$, by simply taking the Difference of the Logarithms of the Heights of the Barometer, or will be equal Log. B — Log. b; and in any other Degree of Heat will be greater or less, in Proportion as the Rarity of the Air is greater or less than the fixed Temperature; or greater or less by $\frac{1}{113}$ th Part of the Whole, for every Degree of the Thermometer reckoned from the fixed Temperature 16 $\frac{1}{2}$; and consequently the Height of the Place will be expressed generally, in French Toises,

$$\text{by this Formula, } \text{Log. B} - \text{Log. b} + \frac{\text{Log. B} - \text{Log. b} \times C - 16\frac{1}{2}}{215} = \text{Log. B} - \text{Log. b} \times 1 + \frac{C - 16\frac{1}{2}}{215}.$$

To reduce this Formula to English Measure, and to the Scale of Fahrenheit's Thermometer, we should first premise some Particulars. The French Foot is to the English Foot as 1.06575 to 1, as was found by a very accurate Experiment: See *Phil. Trans.* Vol. LVIII. for 1768, P. 326: And it is well known, that the Point of freezing, on Fahrenheit's Thermometer, is at 32, and that of boiling Water at 212, or the Interval between them 180 Degrees. But M. De Luc's Point of boiling Water, 80, was marked when the Barometer was at 27 French Inches; and it is the Custom of our principal English Workmen to mark the Point of boiling Water, 212, on Fahrenheit's Thermometer, when the Barometer stands at 30 Inches, which is equal to 28 Inches 1.8 Lines French Measure; or 13.8 Lines higher than M. De Luc's Barometer, when he set off the Point of boiling Water on his Thermometers; and it is well known, that the Heat of boiling Water varies with the

the Weight of the Atmosphere. M. De Luc finds, by his Experiments, this Rule, that an Increase of 1 Line, in the Height of the Barometer, raises the Quicksilver of the Thermometer, placed in boiling Water, by $\frac{1}{114}$ th Part of the Interval between the freezing Point and that of boiling Water: He afterwards, indeed, found that this Rule would not answer for such large Variations of the Barometer as take Place in ascending to very great Heights above the Earth's Surface; (*vid. Essai sur les variations du chaleur de l'eau bouillie*;) but it is accurate enough for any small Variation of the Barometer, on one Side or other of its mean Height, in these lowest Regions of the Atmosphere. The Change, therefore, of the Boiling-Point, on *Fahrenheit's* Scale, for a Change of 1 Line in the Barometer, will be $\frac{1.80}{114} = 0.16$; therefore 13.8 Lines will cause $0.16 \times 13.8 = 2.2$ Degrees of *Fahrenheit's* Scale; and a Thermometer, whose Point of boiling Water was marked 212, when the Barometer stood at 30 *English* Inches, (= 28 Inches 1.8 Line *French* Measure,) will, when the Barometer descends to 27 *French* Inches, sink 2.2 Degrees in boiling Water, or to 209.8, or, in round Numbers, to 210 Degrees, which is distant only 178 from 32, the Point of freezing. Hence, an Extent of 80 d. of M. De Luc's Thermometer, answers to an Extent of 178 of our *Fahrenheit's* Thermometer: and, putting F for the Degrees of this Thermometer, (corresponding to C of M. De Luc's,) we shall have, C : F —

32 :: 80 : 178, and $C = F - 32 \times \frac{80}{178}$, which, substituted in

M. De Luc's Formula, gives $\text{Log. } B - \text{Log. } b \times 1 + \frac{C - 16\frac{1}{2}}{215}$

$$= \text{Log. } B - \text{Log. } b \times 1 + \frac{F - 32 \times \frac{80}{178} - 16\frac{1}{2}}{215} = \text{Log. } B - \text{Log. } b$$

$$\times 1 + \frac{80}{178 \times 215} F - 32 - \frac{170}{178} 16.75 = \text{Log. } B - \text{Log. } b \times$$

$$1 + \frac{F - 32 - 27.27}{478.38} = \text{Log. } B - \text{Log. } b \times 1 + \frac{F - 69.27}{478.38}$$

Where the Answer will still come out in *French Toises*, though adapted to *Fahrenheit's* Thermometer. To bring it out in *English Fathoms*, (or Measure of 6 Feet,) multiply the above Expression

by 1.06575, and we shall have $\text{Logarithm } B - \text{Logarithm } b \times$

$$1 + \frac{F - 69.27}{478.38} \times 1.06575 = \text{Log. } B - \text{Log. } b \times 409.11 + F$$

$$\times \frac{1.06575}{478.38} = \text{Log. } B - \text{Log. } b \times \frac{409.11 + F}{448.87}, \text{ or, in round}$$

$$\text{Numbers,} = \frac{\text{Log. B} - \text{Log. } b \times \frac{409 + F}{449}}{\times 1 + \frac{F - 40}{449}}$$

which will express the Height between the two

Stations in English Fathoms.

123. In the foregoing Expressions, *B* and *b*, as has been mentioned before, signify Heights of the Barometer at the lower and higher Stations, both corrected according to M. De Luc's Directions, for the Difference of Heat between a fixed Temperature (namely, 1st of the Interval between freezing and boiling Water) and the present Heat, indicated by the Thermometer attached to the Barometer at each Station; but it is not necessary to correct both Barometers for the Effect of Heat, but only one, for the Difference of Heat of the two; which will be more convenient, also, on another Account, because the Difference of Heat, at the two Stations, will be generally small, and the Correction, to reduce one Barometer to the Heat of the other, will consequently be small also; whereas, the Difference of the present Heat and the fixed Temperature, and consequently the Correction of both Barometers, may be frequently very considerable. This is evident: Because, if the Heat of the Barometers, at both Stations, was the same, (however different from the fixed Temperature chosen by M. De Luc,) no Correction would be necessary; the Mercury in the Barometer, in both Stations, being expanded in the same Proportion, and consequently the Difference of the Logarithms of its Height, at both Stations, being the same as if the Heat of both Barometers had agreed with that of the fixed Temperature. I shall now, therefore, suppose the upper Barometer is to be corrected, to reduce it to the Temperature of the lower one, and that *b* signifies the Height of this Barometer, as observed, and not yet corrected: The Correction, from what has been said above, calling *D* the Difference of Height of the Thermometer attached to the Barometer at the

two Stations, will be $\pm \frac{D b}{54 K}$, according as the Thermometer

stands highest at the lower or upper Station; and the upper

Barometer, corrected, instead of *b*, will be $b \pm \frac{D b}{54 K}$, which,

substituted in the Formula, gives $\text{Log. B} - \text{Log. } (b \pm \frac{D b}{54 K}) \times$

$\times 1 + \frac{F - 40}{449}$. But the Correction, on Account of the Differ-

ence of Heat of the Barometer at the two Stations, may be reduced

duced to a still easier Expression, in which the variable Quantity, b , the Height of the upper Barometer, shall not appear. The Fluxion of a Logarithm is, to the Fluxion of its natural Number, as the Modulus of the System to the natural Number; and, 4343 is the Modulus of the common Logarithms, when the four Places, next following the Characteristic, are taken as whole Numbers instead of Decimals, which is meant to be done in the

Use of the foregoing Formula. Therefore $\frac{D b}{54 K}$ being very small, with Respect to b , we shall have, Variation of Log. b :

Variation of b ($= \frac{D b}{54 K}$) $:: 4343 : b$, very nearly; and thence

Variation of Log. $b = \pm \frac{D b}{54 K} \times \frac{4343}{b} = \pm \frac{4343 D}{54 K}$. Which

(putting $K = 178$) $= \pm 0.452 D$. Hence Log. ($b \pm \frac{D b}{54 K}$)

$= \text{Log. } b \pm 0.452 D$; which, being substituted in the Formula above, will give the Difference of Height of the two Stations in *English* Fathoms, in a more convenient Expression; namely,

Log. $B - \text{Log. } b \mp 0.452 D \times 1 + \frac{F - 40}{449}$; where the upper

Sign, —, is to be used when the Thermometer of the Barometer is highest at the lower Station, and the lower Sign, +, is to be used when the said Thermometer is lowest at the lower Station. The first Case will be most common; especially where the Difference of Height of the two Stations is considerable. It should also be observed, that when F , the Height of *Fahrenheit's*

Thermometer, is less than 40 d. $+\frac{F - 40}{449}$ becoming negative or subtractive, it must be applied in the Calculation accordingly.

124. It may perhaps be convenient to repeat here the Meaning of the algebraic Terms used in the foregoing Formula, that any Person may make Use of it, without having Occasion to recur to the foregoing Investigation. B signifies the observed Altitude of the Barometer at the lower Station, and b that at the upper Station; Log. B and Log. b signify their Logarithms, taken out of the common Tables, by assuming the four first Figures next following the Characteristic as whole Numbers, and considering the three remaining Figures, to the Right-Hand, as Decimals: D signifies the Difference of Height of *Fahrenheit's* Thermometer, attached to the Barometer at the Top and Bottom of the Hill: and F signifies the Mean of the two Heights of

Fahrenheit's Thermometer, exposed freely for a few Minutes to the open Air in the Shade, at the Top and Bottom of the Hill.

125. The *Formula* for the Measure of Heights may also be changed, and adapted to Thermometers of particular Scales, for the Convenience of Calculation, as *M. De Luc* has done; but these Scales will be different from his. The Thermometer, attached to the Barometer, had better be divided with the Interval between freezing and boiling Water, consisting of 81.4 d. ($= 180 \times .452$); the Freezing-Point may be marked 0, and the Point of boiling Water will be 81.4; for, then, if the Difference of Height of this Thermometer, at the two Stations, be called d , we shall have $d = 0.452 \times D$; for $d : D :: 81.4 : 180 :: 0.452 : 1$, and the Number of Degrees, expressed by d , will shew immediately the Correction for the Difference of Heat of the two Barometers. If the Thermometer, designed to shew the Temperature of the Air, be divided with the Interval between freezing and boiling Water $= 200$, and the Freezing-Point be marked -9 , and the Boiling-Point $+191$, and the Heights of this Thermometer, at the two Stations, be called

G and I, we shall have $\frac{F-40}{449} = \frac{G+I}{2 \times 500} = \frac{G+I}{1000}$. For

$F-40 = F-32-8$ is the Height of *Fahrenheit's* Thermometer, reckoned from 8 Degrees above freezing, and $449 : 500 :: 180$

$: 200 :: 8 : 9$; and the Fraction $\frac{F-32-8}{449}$, if both the

Numerator and Denominator be increased in the Ratio of 449 to

500, will become $= \frac{F-32-8 \times \frac{500}{449}}{500} = \frac{F-32 \frac{500}{449} - 9}{500}$

$= \frac{G+I}{2 \times 500} = \frac{G+I}{1000}$, because $\frac{G+I}{2} + 9 = F-32 \times \frac{500}{449}$.

Therefore, if the Thermometer of the Barometer has the freezing-Point marked 0, and the Point of boiling Water 81.4, and the Difference of its Height, at the two Stations, be called d ; and if the Thermometer, for measuring the Temperature of the Air, be divided with the Interval of 200, between the Freezing-Point and that of boiling Water, and the first be marked -9 and the latter $+191$, and the Degrees, shewn by this at the two Stations, be called G and I; the *Formula*, that will give the Height of the upper Station above the lower one, in *English* Fa-

thoms, will be $\text{Log. B} - \text{Log. b} \mp d \times 1 + \frac{G+I}{1000}$; which

consequently, multiplied by 6, will give the Height in *English* Feet. It is to be observed, as before, that $-d$ or $+d$ is to be used, according as the Thermometer, attached to the Barometer,

is highest at the lower or upper Station ; and, if G and I should happen to fall below 0 of the Scale, or to be subtractive, they must be applied accordingly in the Calculation.

126. I shall now add Nothing more, but give the Rule for finding Heights by the Barometer, according to the *Formula* delivered above, in common Language ; first, as adapted to *Fahrenheit's* Thermometer, and, next, as adapted to the two Thermometers of particular Scales. Take the Difference of the tabular Logarithms of the observed Heights of the Barometer, at the two Stations, considering the 4 first Figures, exclusive of the Index, as whole Numbers, and the 3 remaining Figures, to the Right, as Decimals, and subtract or add $\frac{4.32}{10000}$ th of the Difference of the Altitude of *Fahrenheit's* Thermometer, attached to the Barometer at the two Stations, according as it was highest at the lower or upper Station ; thus you will have the Height of the upper Station above the lower, in *English* Fathoms, nearly ; which is to be corrected as follows : Make this Proportion : As 449 is to the Difference of the mean Altitude of *Fahrenheit's* Thermometer, (exposed to the Air at the two Stations, from 40° ,) so is the Height of the upper Station, found nearly, to the Correction of the same ; which added or subtracted, according as the mean Altitude of *Fahrenheit's* Thermometer was higher or lower than 40° , will give the true Height of the upper Station above the lower, in *English* Fathoms ; and, multiplied by 6, will give it in *English* Feet.

127. The same Rule, adapted to the Thermometers of particular Scales, is this :

Take the Difference of the tabular Logarithms of the observed Heights of the Barometer, at the two Stations, considering the 4 first Figures (exclusive of the Index) as whole Numbers, and the 3 remaining Figures, to the Right, as Decimals ; and subtract or add the Difference of a Thermometer of a particular Scale, attached to the Barometer at the two Stations,

Stations, according as it was highest at the lower or upper Station, and you will have the Height of the upper Station above the lower one, in *English Fathoms*, nearly; which is to be corrected as follows: Make this Proportion: As 1000 is to the Sum of the Altitudes of a Thermometer of a particular Scale, exposed to the Air at both Stations, so is the Height of the upper Station above the lower, found nearly, to the Correction of the same; which added or subtracted, according as the Sum of the Altitudes of the Thermometer, exposed to the Air, is positive or negative, will give the true Height of the upper Station above the lower, in *English Fathoms*; and, multiplied by 6, will give it in *English Feet*.

NEVIL MASKELYNE.

128. As we could not abridge this Paper without injuring it, we doubt not but that the ingenious Author would rather see it printed here entire, *Samuel Horsley, L. L. D.* has also, in the same Transactions, a very curious Paper on this Subject, on *M. De Luc's* Principles; but Room will only permit us to give some of the Tables he has calculated, (to render the Practice easier,) and to illustrate their Use in a different Manner, which, perhaps, may be more easily understood by the practical Surveyor than that of the learned Author.

129. The Heights of Hills may be easily computed, by Means of the following Tables, by these Rules: First, take out the Logarithms of the Heights of the Barometer, at Top and Bottom of the Hill, from a Table which has them to 7 decimal Places; but the Index and all the decimal Places are here to be considered as whole Numbers. Subtract the lesser Logarithm from the greater, and divide the Remainder by 1000, which is readily done, by cutting off 3 Figures on the Right-Hand; the Quotient will give a Number of *English Fathoms*, (each 6 Feet,) which we will call the Gross-Height.

EXAMPLE.

EXAMPLE. Let the lower Barometer be supposed to stand at 29.68 Inches, the upper at 25.28 Inches:

The Log. of 29.68 = 14724639 } taken as whole
Log. of 25.28 = 14027771 } Numbers.

Their Diff. $\div$ by 1000 = 696.868 = Gross-Height
[of the Hill, in Fathoms.

N. B. What is here called the *Gross-Height* of the Hill is the *true Height*, when the Thermometer stands at 40° both at Top and Bottom of the Hill; but, if not, it will require two Corrections, as will be explained in the following Rules.

130. Secondly, With the Difference of the Temperatures of the Quicksilver, (as shewn by the Thermometers in the Cases of your portable Barometers,) take out of Table I. from the Column of Fathoms, the Equation corresponding thereto, which call the *Equation of the Mercury*: But, if the Thermometer, in the higher Station, gives a warmer Air than in the lower Station, then we must add this Equation to, otherwise subtract it from, the Gross-Height, and the Sum, or Remainder, will be the *Approximate-Height*.

EXAMPLE. Let the Gross-Height of a Hill, found by the last Article, be 696.868, and let us suppose the Thermometer, at the Bottom of the Hill, stood at +57, and, at the Top of the Hill, at +43, their Difference is 14.

Against 10, in the 3d Column of Table I. is 4.518

4 ————— 1.807

The Sum is the Equation of the Mercury 6.325

From the Gross-Height of the Hill 696.868

Subtract the Equation of Mercury 6.325

Gives the Approximate-Height 690.543

131. Thirdly, Add together the Temperatures of the Air, shewn by the Thermometers (in open Air) under their proper Signs, and call Half the Sum the *Temperature of the Air*, being the mean State of the Air. With this Temperature, from Table II. find Part of the Equation for the Air thus: Look at Top for the Number of Decades (or Tens) next less than the Temperature of the Air; and from that Column collect (by repeated Entries, if necessary) the Corrections for the Units, Tens, and Hundreds, of Fathoms, in the Approximate-Height: (And, if you should want for Thousands of Fathoms, it is only removing the Decimal-Point in Hundreds of Fathoms, by the Rules of Decimals:) Find the Sum of these several Parts of the Correction, and call it N.

To find that Part of this Equation, called n , which corresponds to the Degrees in the Temperature of the Air, which are over and above the Decades, observe, that the Correction for 1 Fathom, at the several Degrees, is here annexed.

Degrees.	1	2	3	4	5	6	7	8	9
Equation for 1 Fathom.	.002	.004	.007	.009	.011	.013	.015	.018	.020

Therefore, to find n , we have only to multiply the Equation for 1 Fathom, taken from this little Table, by the Number of Fathoms in the Approximate-Height, and the Product will give n .

If the Sign of the Correction, N, found by Table II. is +, add the Sum $N+n$ to the Approximate-Height; but, if the Sign of Correction, N, is -, subtract their Difference, viz. $N-n$, from the Approximate-Height, for the correct Height of the Object, which was required.

EXAMPLE. Let the Approximate-Height be, as found by the last Article, 690.543, and the Height of Mercury in the Thermometers, out of Doors, 42 and 57; required, the true Height of the Hill.

Thermometers

Altimetry.

65

Thermometers Heights } $\begin{matrix} 42 \\ 57 \end{matrix}$

Sum $\underline{99}$

The Temperature of Air $49\frac{1}{2} = 40^{\circ} + 9\frac{1}{2} \text{ d.}$

In Table II. in Column 40+, the Equation is 0; therefore $N = +0$. In the little Table, in this Art. the Equation for 1 Fathom, corresponding to 1 Degree, is $= .002$; and therefore to $\frac{1}{2}$ a deg. $= .001$
Under 9 d. $= .020$

Hence $9\frac{1}{2} \text{ d.} = .021$ this

$\times$ by Number of Fathoms, 690, gives $14.49 = n$.
 $\therefore$ the whole Equation $N + n = 0 + 14.49 = 14.49$ Fathoms.

Fathoms.

To the approximate Height 690.543

Add Equation for the Air $N + n = 14.490$

The Sum is the required Height $= 705.033$

132. EXAMPLE 2. Let the Heights of the Mercury in the Barometers and Thermometers be as under:

Barometer.	Therm. in.	Diff.	Therm. out.	Temp. of Air.
Lower 29.45	+38	3	+31	33
Upper 26.82	+41		+35	

Log. of 29.45 14690853

Log. of upper 26.82 14295908

Their Diff. divided by 1000 $= 394.945$ the gross
Diff. of Therm. in being 3 Deg. } Height in
gives Equation of Mercury, } $+ 1.356$ Fathoms.
by Table I. }

Approximate Height 396.301 Fathoms.
K Now,

Altimetry.

Now, to find N , we have given the Temperature of the Air $= 33^\circ = 30^\circ + 3^\circ$.

By Table II. 30d. upon 300 Fathoms give 6.683
 Upon 90 2.005
 Upon 6 0.134

Sum is $N = 8.822$

To find n . By the little Table in Art. 131, under 3 Degrees, we have for 1 Fathom .007, this $\times d$ by 396 gives $n = 2.772$. Now, N being — and n always +, the Equation is the Difference of these Numbers, (*viz.*) $8.822 - 2.772 = 6.050$.

From the approximate Height 396.301 Fath.
 Subtract Equation for Air $N - n = 6.050$

Gives the correct Height in Fathoms 390.251

133. EXAMPLE 3. Let the Heights of the Barometers be 30.0 and 29.9 Inches, and the Heights of the Thermometers each $+40$ of *Fahrenheit's* Scale.

Log. of 30.0 = 14771212

Log. of 29.9 = 14756712

Their Diff. $\div$ by 1000 = 14500 Fathoms; this, multiplied by 6, gives 87 Feet; the Height answering to $\frac{1}{10}$ of an Inch Fall of the Mercury, in a mean State of the Air; *viz.* when the Thermometer stands at 40 d. and the Barometer at 30 Inches.

134. TABLE I.

Deg. of Bird's Fahrenheit.	Correction of Dif. of Log.	Fath. or $\frac{1}{1000}$ Parts of Dif. of Log.
1	452	0.452
2	904	0.904
3	1355	1.355
4	1807	1.807
5	2259	2.259
6	2711	2.711
7	3163	3.163
8	3614	3.614
9	4066	4.066
10	4518	4.518
20	9036	9.036
30	13554	13.554
40	18072	18.072
50	22590	22.590
60	27108	27.108
70	31626	31.626

TABLE II.

Deg.	40+	50+	60+	70+	80+
	30—	20—	10—	0—	
1	0	0.022	0.044	0.067	0.089
2	0	0.044	0.089	0.134	0.178
3	0	0.067	0.134	0.200	0.267
4	0	0.089	0.178	0.267	0.356
5	0	0.111	0.223	0.334	0.445
6	0	0.134	0.267	0.401	0.535
7	0	0.156	0.312	0.468	0.624
8	0	0.178	0.356	0.535	0.713
9	0	0.200	0.401	0.601	0.802
10	0	0.223	0.445	0.668	0.891
20	0	0.445	0.891	1.337	1.782
30	0	0.668	1.337	2.005	2.673
40	0	0.891	1.782	2.673	3.564
50	0	1.114	2.228	3.342	4.455
60	0	1.337	2.673	4.010	5.347
70	0	1.559	3.119	4.678	6.238
80	0	1.782	3.564	5.347	7.129
90	0	2.005	4.010	6.015	8.020
100	0	2.228	4.455	6.683	8.911
200	0	4.455	8.911	13.367	17.822
300	0	6.683	13.367	20.050	26.733
400	0	8.911	17.822	26.733	35.645
500	0	11.139	22.278	33.417	44.556
600	0	13.367	26.733	40.100	53.467
700	0	15.594	31.189	46.783	62.378
800	0	17.822	35.645	53.467	71.289
900	0	20.050	40.100	60.150	80.200

BOOK III. CHAP. II.

Longimetry.

135. **L**ONGIMETRY shews how to apply the Doctrine of Triangles to find the Distances of Objects.

Prop. 1. Standing on an Object of a known Height, to find its Distance from another Object.

K 2

EXAMPLE.

Pl. V.
Fig. 5.

EXAMPLE. The Eye being at the Window of the House C, 20 Feet above the horizontal Plane AB, and the $\angle$ of Depression ACD (DC being conceived to be $\parallel$ to AB) 25 d. required the Distance of the Tree from the House, viz. AB.

Construction. Make $\angle BAC = 25$ d. (for $\angle BAC = \angle DCA$, by the Nature of $\parallel$ s,) and construct the Δ as in Case 3 of right-angled Δ s. Then, by that Case, we have,

Calculation. As Sine of $\angle A$ 25 d. : BC 20 Feet :: Cosine of $\angle A$: AB = 42.89 Feet. Q. E. I.

136. *Scholium 1.* In a similar Manner may be found the Depth of a Well, which is of the same Bigness at Top and Bottom. For, measuring AB, the Width of the Well, and taking the $\angle$ of Depression BAC, we have, As Cosine of $\angle CAB$: AB :: S. $\angle CAB$: BC, the Depth which was required.

Pl. V.
Fig. 6.

137. *Scholium 2.* In Order to take the horizontal $\angle$ s, the Quadrant may be supported by a Ball and Socket, fixed to the Middle of the under Side, and the Whole supported by the Legs of a common Surveying-Table. But as, in Practice, it may frequently happen, that the $\angle$ to be taken may be obtuse, it will be better to make Use of a Semicircle, instead of a Quadrant, which may be thus made.

Pl. V.
Fig. 7.

Let ADB be a Semicircle of Brass, of a Radius sufficient to be divided into Degrees and Minutes; which being carefully done, and numbered both Ways, viz. from A towards B, and from B towards A, (by the Method shewn in Geometry, Art. 211,) on the Center C place a strait Rule to move round the Center, and at the End E a perpendicular open Sight, with a vertical Catgut-String or Horse-Hair; and at the End F an Eye Sight, having a vertical Slit for the Eye to look through: At G may be placed a Compass; but, if the Position with Respect to the Meridian be not required, and it is only required to find the Distances of Objects, the Compass is not absolutely necessary.

138. This Instrument being properly supported on the three-legged Staff, the Manner of using it, to take an $\angle$, is very easy: *Viz.* if it is required to take an $\angle$ ICH, lay the Index, or Rule, so that the middle Line of the Index may correspond with the Line AB, and through the Sight you see the Object I; then, turning the Index about till, through the Sights, you see the Object H, the Number of Degrees, in the Arc BE, is manifestly $\equiv$ the $\angle$ required. In like Manner, the oblique Angle KCH may be taken, and the Number of Degrees, in the Arc AE, shews the required $\angle$.

139. If the Objects are at a considerable Distance, it would be proper to have a Telescope instead of plain Sights, also a Spirit-Level, &c. But as, when we treat of Surveying, we shall be obliged fully to describe a Theodolite, an Instrument much better adapted to this Purpose, when properly constructed, we judge it improper to say any Thing farther on this Subject here, and therefore pass on to

140. *Proposition 2.* To find the Distance of two Objects, which are inaccessible from each other, but both accessible from another Place, A.

EXAMPLE. Wanting to find the Distance of two Towers, B, C, at a convenient Distance A. I took the $\angle$ contained BAC $\equiv$ 40 d. then I measured AC, which was 2.72 Miles, and AB 2.5 Miles: Required the Distance asunder of the Towers, B, C. Pl. V.
Fig. 8.

This is the same as Case 3 of oblique Triangles; by which BC will be found $\equiv$ 1.797 Mile.

141. *Proposition 3.* To find the Distance of an inaccessible Object, or Breadth of a River.

EXAMPLE. Wanting to find the Breadth of a River, I observed a Tree close to the other Side: At a convenient Station A, by the Bank of the River, I took the $\angle$ CAB $\equiv$ 50 d. then measuring AB $\equiv$ 100 Yards, I took the $\angle$ CBA $\equiv$ 60 d. Quære the Breadth of the River. Pl. V.
Fig. 9.

Construction.

Construction. Draw $AB = 100$ Yards, make $\angle A = 50^\circ$ d. and $\angle B = 60^\circ$ d. then is the Place of Intersection, C, that of the Tree on the opposite Bank of the River.

Calculation. First $50^\circ + 60^\circ = 110^\circ$, and $180^\circ - 110^\circ = 70^\circ = \angle C$. Then, As $S. \angle C 70^\circ : AB \ 100$ Yards $:: S. \angle B 60^\circ : AC \ 92.16$ Yards, the Distance of the Object C from the Place A, required.

Again, letting fall the $\perp CD$, we have, in the right-angled $\triangle ADC$, As Radius $90^\circ : AC \ 92.16$ Yards, $:: S. \angle A \ 50^\circ :: DC = 70.59$ Yards. *Q. E. I.*

142. *Scholium.* By this Proposition it is that the Trees, where Bees hive, are discovered in *New-England*. For, in the *Philosophical Transactions*, Number 367, Jan. 1721, Paul Dudley, Esq. remarks to this Purport. The Hunter, in a clear sun-shiny Day, takes a Plate or Trencher, with a little Sugar, Honey, or Melasses, spread on it, and, when got into the Woods, sets it down on a Rock or Stump there; this the Bees soon find out; for it is generally supposed a Bee will scent Honey or Wax above a Mile's Distance; and it is observed, when one Bee goes Home from the Sugar-Plate, he returns with a considerable Number from the Hive. The Hunter secures in a Box, or other Conveniency, some of them, as they fill themselves, and, after a little Time, lets one of them go, observing very carefully the Course it steers, by a pocket Compass; for, after it rises in the Air, it flies directly to the Tree where the Hive is. Then the Hunter walks away (as nearly as he can at Right-Angles to the Course of the Bee) a convenient Distance, which he measures; then he lets another Bee fly away, and observes, by the Compass, his Course towards the Tree also: Hence he knows, by this Proposition, the Distance the Tree is from him, and, consequently, the Course being also known, he can find whereabouts the Tree is, and so come at the Honey.

143. *Proposition 4.* To find the Distance of two inaccessible Objects.

EXAMPLE. Wanting to find the Distance of two Churches, separated from me by a River, I fix on two convenient Stations A, B: Then, at A, I take $\angle BAD = 110^\circ$ and $\angle BAC = 59^\circ$, then measuring $AB = 1$ Mile, at B, I take the $\angle ABC = 106^\circ$ and $\angle ABD = 54^\circ 30'$.

Pl. V.
Fig. 10.

Construction. Draw $AB = 1$ Mile; make the $\angle$ s as in the Example; then will the Places of Intersection, D and C, be the Places of the two Churches, whose Distance asunder is required.

Calculation. $\angle DAB 110^\circ + \angle ABD 54^\circ 30' = 164^\circ 30'$, and $180^\circ - 164^\circ 30' = 15^\circ 30' = \angle ADB$. Now, as $S. \angle ADB 15^\circ 30' : AB 1 \text{ Mile} :: S. \angle ABD 54^\circ 30' : AD 3.046 \text{ Miles}$. Again, the $\angle ABC 106^\circ + \angle BAC 59^\circ = 165^\circ$. $\therefore \angle ACB = 180^\circ - 165^\circ = 15^\circ$. Hence, by Trigonometry, As $S. \angle ACB 15^\circ : AB 1 \text{ Mile} :: S. \angle ABC 106^\circ : AC 3.714 \text{ Miles}$. Now, in the Triangle DAC, we have given $DA = 3.046 \text{ Miles}$, $CA = 3.714 \text{ Miles}$, and the $\angle$ contained $DAC (= \angle DAB 110^\circ - \angle BAC 59^\circ =) 51^\circ$; which is the 3d Case of oblique Triangles, by which DC will be found $= 2.971 \text{ Miles}$. Q. E. I.

144. In the same Manner the Distances of three or four Objects may be found. When, by these Means, the Distance of two Towers or Objects is found, that may be taken for a new Base; and, by such Means, others may be found, and so a Chain of Triangles formed; by which Means, if the $\angle$ s are carefully taken, by Instruments accurately divided, the Situation of the principal Towers, Hills, &c. in a Country, may be taken.

145. *Proposition 5.* To find the Distance and Position of a Place from three other Places, whose Position and Distances from each other are given.

This Proposition admits of 6 Cases; most of which are of frequent and great Use in Surveying, as will appear by the Examples.

Case

Case 1. When the Station is without the Triangle made by the three given Objects, but in a Line with one of the Sides produced.

EXAMPLE. Having, in a Map of an Estate, found three Trees, A, C, B, planned, but the Tree D in a direct Line with the Trees A, C, I found, from A to C measured 8.90 Chains, A to B 12.20, and B to C 7.70; and that the $\angle CDB$ was 27° . Quære the Distance AD, in Order to set the Tree in its proper Place in the Map.

Construction. The $\triangle ABC$ being formed, (and the $\angle BAC$ being $\ast =$ the two internal and opposite $\angle$ s D and DBA) make the $\angle DBA =$ the Difference of the $\angle$ s BAC and BDA, then will BD cut CA, produced, in D, the Point required.

Calculation. The three Sides of the Triangle ACB being given, find the $\angle BAC$ by Case 4 of oblique Triangles; which, in the above Example, will be found $= 39^\circ 0'$: Then $\angle BAC 39^\circ 0' - \angle ADB 27^\circ = 12^\circ = \angle DBA$, for the Reason given in the Construction. Hence, in $\triangle DAB$, we have, As $S. \angle D : AB :: S. \angle DBA : DA$, $=$, in this Example, 5.587 Chains. Q. E. I.

146. *Case 2.* When the Station required is in one of the Sides of the given $\triangle$.

EXAMPLE. Let the $\triangle ABC$ be the same as in the first Example: And, having Occasion to lay down an Object at D, I took the $\angle BDC = 73^\circ$. Quære the Distance DB or DA.

Construction. Make $\angle BAE =$ the observed $\angle BDC$, in our Example, $= 73^\circ$ d. draw $CD \parallel EA$; then is D manifestly the Place required.

Calculation. In the $\triangle ABC$, by the 4th Case of oblique Triangles, find the $\angle B$; then, the $\angle$ s B and BDC being known, the $\angle DCB$ is also known. Hence we have, As $S. \angle BDC : BC :: S. \angle DCB : BD$, equal, in the above Example, to 6.996 Chains.

147. *Case 3.* When the three given Places are in a Right-Line.

EXAMPLE.

EXAMPLE. Being at Sea, near a strait Shore, I observed three Objects, A, B, C, which were truly laid down on my Chart; but, wanting to lay down the Place of a sunken Rock, D, by *Hadley's** *Octant*, I took the $\angle ADB = 26^{\circ} 15'$ and $\angle BDC = 23^{\circ} 30'$, and, by measuring on the Map, found $AB = 2.2$ Miles, $BC = 1.55$ Mile.

Construction. On AB describe the Segment of a Circle, ADB, capable of containing an $\angle = 26^{\circ} 15'$, and, on BC, the Segment BDC, capable of containing an $\angle = 23^{\circ} 30'$: Then, as the Point D must be somewhere in the Arc of the Segment ADB, and also somewhere in the Arc of the Segment CDB, it must be where they intersect in D: Therefore, joining D, A, D, B, and D, C, it is manifest DA, DB, and DC, are the Distances required.

Calculation. Having drawn the Radii AE, ED, EB, also the Radii FD, FB, FC, from the Middle of BD, viz. G, draw Lines GE, GF: Now, the 3 Sides of the Triangles EDG, EBG, being respectively equal, (viz. $ED = EB$, $BG = DG$, by Construction, and EG common,) these Δ s must be equal to each other in every Respect; and $\therefore \angle EGD = \angle EGB$; and $\therefore$ each a Right- $\angle$. For the like Reason, the $\angle DGF$ is a Right- $\angle$; and, consequently, EGF, joining the Centers of the two Circles, is a Right-Line. Let fall the $\perp$ s EI, FH; then, the $\angle$ at the Center being double the $\angle$ at the Circumference, the $\angle AEB =$ twice the $\angle ADB$, $\therefore$ Half the $\angle AEB$, viz. $\angle IEB = \angle ADB = 26^{\circ} 15'$; $\therefore \angle IBE = 90^{\circ} - \angle IEB = 26^{\circ} 15' = 63^{\circ} 45'$; and, for the same Reason, $\angle BFH = \angle BDC = 23^{\circ} 30'$, and so $\angle FBH = 90^{\circ} - 23^{\circ} 30' = 66^{\circ} 30'$; but the $\angle IBE + \angle EBF + \angle FBH = 180^{\circ}$. $\therefore \angle EBF = 180^{\circ} - \angle IBE - \angle FBH = 180^{\circ} - 63^{\circ} 45' - 66^{\circ} 30' = 49^{\circ} 45'$.

In the right- $\angle \Delta$ IBE are now given IB = Half AB = 1.1 Mile; and $\angle$ s to find EB, the Radius of the

* The Manner of taking such Angles, by *Hadley's* *Octant*, will be shewn in our *Elements of Navigation*, which are nearly ready for the Press.

the Circle ABD, ≈ 2.487 Miles. Also, in the right-
 $\angle$ d $\triangle BFH$ are given $BH = \text{Half } BC = 0.775$, and
 $\angle s$ to find BF , the Radius of the Circle BCD, $\approx$
 1.944 .

Hence, in the $\triangle EBF$, we have now given $EB =$
 2.487 , $BF \approx 1.944$, and contained $\angle EBF = 49 \text{ d. } 45'$
(as found above) to find the $\angle BEF = 50 \text{ d. } 19'$.

Then, in the right- $\angle$ $\triangle BEG$, are given BE and
 BEG to find $BG \approx 1.914$ Miles. But $BG = \text{Half}$
 BD by Supposition; $\therefore BD = \text{twice } BG = 3.828$
Miles.

Again, the $\angle BAD$ being at the Circumference,
and the $\angle BED$ at the Center, of the same Circle
BAD, and standing on the same Base BD , the $\angle$
 $\bullet 145. g.$ BAD is $\approx \text{Half}^*$ the $\angle BED$, viz. $= \angle BEG = 50 \text{ d.}$
 $19'$ above found. Hence, in the $\triangle BAD$ are given
 $\angle s$ A and D , and consequently the $\angle B = 103 \text{ d. } 26'$,
also the Side AB or BD to find the Side $AD = 4.838$.

Lastly, in the Triangle ADC are given the Sides
 AD , AC , ($= AB + BC$), and $\angle CAD$ (BAD) to find
 $DC \approx 3.781$ Miles.

Or thus.

148. *Construction.* Make $\angle ACE = \angle ADB =$
Pl. V. $26 \text{ d. } 15'$, and $\angle CAE = \angle BDC = 23 \text{ d. } 30'$; and,
Fig. 15. from the Point of Intersection E , through B , draw a
Line to ED to intersect the Arc ADC ; join A, D ,
and D, C ; then are DA, DB, DC , the required
Distances.

Demonstration. For the $\angle s$ ACE and ADE (ADB)
stand on the same Base, AE , in the same Segment
 $ADCE$, and are therefore equal, by Art. 146 of our
Geometry. Also the $\angle EAC$ and $\angle EDC$ are equal,
being on the same Base, EC , of the same Segment
 $EADC$.

Calculation. In Triangle AEC are given $\angle s$ EAC ,
 ECA , and consequently $\angle AEC$; also AC to find
 $AE \approx 2.173$ Miles. Then, in $\triangle ABE$ will be given
two Sides AB, AE , and contained $\angle BAE = 23 \text{ d.}$
 $30'$, to find $\angle s$ $AEB \approx 79 \text{ d. } 57'$ $ABE \approx 76 \text{ d. } 33'$.

But

But $\angle DBC = \angle ABE$, by Art. 69, *Geometry*. Hence, in $\triangle BDC$ are given $\angle D$ and $\angle DBC$, and consequently $\angle BCD$ is also known; also BC is given by the Question, whence may be found $BD = 3.828$ and $DC = 3.781$.

Lastly, in $\triangle AED$ are given $\angle s$ ADE , AED , and Side AE , to find $AD = 4.838$.

149. *Case 4.* When the three Objects form a Triangle, and the Station is without the Triangle.

EXAMPLE. Wanting to determine the Position of a Rock, at a Distance from the Shore or main Land, by *Hadley's* Octant, I took the $\angle ADC = 54$ d. $\angle CDB = 33$ d. and, by measuring on an actual Survey of the Coast, I found $AC = 2.27$ Miles, $BC = 1.55$ Miles, and $AB = 3.3$ Miles. Hence it is required to find the Distance of each of the 3 Places from the Station D .

Pl. VI.
Fig. 1.

The *Construction* is nearly the same as in the last; viz. on AC describe a Circle capable of containing an $\angle ADC = 54$ d. and on CB a Segment capable of containing an $\angle CDB$ of 33 d. then is the Point of Intersection, D , the Place required.

Pl. VI.
Fig. 2.

Calculation. Conceive the several Radii, &c. to be drawn as in the Figure: Then, in $\triangle ABC$ are given three Sides to find $\angle ACB = 118$ d. $17'$: And, for the Reason shewn in the last Case, in the right- $\angle$ $\triangle CEF$ are given $EC = \text{Half } AC = 1.135$, and $\angle EFC = \angle ADC = 54$ d. to find the Radius $FC = 1.403$. Also, in the right- $\angle$ $\triangle CHG$ are given $CH = \text{Half } CB = 0.775$, and $\angle CGH = \angle CDB = 33$ d. to find the Radius $GC = 1.423$.

Now, the $\angle ECF = 90$ d. — $\angle EFC 54$ d. = 36 d. and $\angle GCH = 90$ d. — $\angle CGH 33$ d. = 57 d. But $\angle ACB 118$ d. $17'$ — $\angle ECF 36$ d. — $\angle GCH 57$ d. = $\angle FCG = 25$ d. $17'$. Hence, in the $\triangle FCG$ are given the two Sides FC , GC , and $\angle$ contained FCG , to find $\angle FGC = 75$ d. $33'$ and $\angle GFC 79$ d. $10'$. Now, in the right- $\angle$ $\triangle GIC$ are given GC and $\angle IGC$ (FGC) to find $IC = 1.378$, which, doubled, gives $DC = 2.756$. Again, $\angle CGB$ being = twice

$\angle CGH = 66d.$ and $\angle CGD$ being = twice $\angle CGI$
 = twice $75d. 33' = 151d. 16'$, the $\angle DGB = 360d.$
 — $\angle CGB 66d.$ — $\angle CGD 151d. 16' = 142d. 44'$.
 Hence, Sum of the $\angle$ s GDB and $GBD = 180d.$ —
 $\angle DGB 142d. 44' = 37d. 16'$. But $\angle$ s GDB and
 GBD are =; $\therefore$ each = Half of $37d. 16' = 18d. 38'$.
 Now, in $\triangle DGB$ we have, As $S. \angle B 18d. 38'$:
 $GD (GC) 1.423 :: S. \angle DGB 142d. 44' : DB =$
 $2.697.$

In the same Manner, $\angle AFD$ may be found =
 $360d.$ — $\angle AFC 108d.$ — $\angle CFD 158d. 20' =$
 $93d. 40'$, and $180d.$ — $\angle AFD =$ Sum of the $\angle$ s
 FAD and FDA , which, being equal, will each be
 = Half that Sum, *viz.* $43d. 10'$: $\therefore$ in $\triangle AFD$
 we have, *lastly*, As $S. \angle ADF 43d. 10'$: $AF (FC)$
 $:: S. \angle AFD : AD = 2.047.$

Or thus.

150. *Construction.* Make $\angle EBA = \angle ADE 54d.$
 and $\angle BAE = \angle EDB = 33d.$ Then, through
 the Points A, B , and the Intersection E , describe a
 Circle $AEBDA$; through E, C , draw EC and pro-
 duce it to intersect the Circle at D ; join AD, BD ;
 then will the Distances AD, CD, BD , be those re-
 quired. For $\angle$ s BAE and BDE , standing on the
 same Base, BE , in the same Segment $BDAE$, are
 equal, and also $\angle$ s ABE and ADE , standing on the
 same Chord, AE , in the same Segment $ADBE$, are
 equal.

Calculation. First, in $\triangle ABC$ are given three
 Sides to find $\angle BAC = 24d. 26'$. Now, in $\triangle AEB$
 are given $\angle BAE = 33d.$ $\angle ABE = 54d.$ and $\angle$
 $AEB = 93d.$ and Side $AB = 3.3$ to find the Sides
 $AE = 2.674$ and $BE = 1.8$. Then, in $\triangle ACE$
 will be given the two Sides AE and $AC 2.27$, and
 their contained $\angle EAC = \angle BAE 33d.$ — $\angle BAC$
 $24d. 26' = 8d. 34'$, to find $\angle AEC = 38d. 19'$.

Again, in $\triangle AED$ are given $AE = 2.674$, $\angle$
 $AED = 38d. 19'$, $\angle ADE = 54d.$ and consequent-
 ly $\angle DAE = 87d. 41'$ to find $AD = 2.049$. Then

$\angle$

Longimetry.

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$\angle ADE$ 54 d. + $\angle AEC$ 38 d. 19' = 92 d. 19',
and 180 d. — 92 d. 19' = 87 d. 41' = $\angle DAB$.
But $\angle EAD$ 87 d. 41' — $\angle EAC$ 8 d. 34' = $\angle CAD$
= 79 d. 7'. Hence, in $\triangle ACD$ are given $AC \pm$
2.27, $\angle ADC = 54$ d. and $\angle CAD = 79$ d. 7', to
find $DC \pm 2.755$.

Lastly, $\angle AEB$ 93 d. — $\angle AEC$ 38 d. 19' = $\angle$
 $DEB = 54$ d. 41'. Hence, in $\triangle DEB$ are given
 $EB = 1.8$, $\angle DEB = 54$ d. 41', and $\angle EDB \pm$
33 d. to find $BD = 2.696$.

N. B. In this Method of constructing, when the
 $\angle BDC$ is less than the $\angle BAC$, the Point C will be
above the Point E ; but the Calculation will be so
exactly like the above as to require no particular Ex-
planation. When the Points E and C happen to
fall too near together to produce EC towards D with
Certainty, the first Method of Construction will be
more accurate.

151. *Case 5.* When the Point D , or our Station,
falls without the $\triangle ABC$, but the Point C falls
towards D .

EXAMPLE. Let A, B, C , represent three Towers, Pl. VI.
Fig. 4.
whose Distances from each other are known: *Viz.*
 $AB = 3.3$ Miles, $AC = 2.27$ Miles, and $BC =$
1.55 Miles. On another Tower, D , I took the $\angle$
 $ADC = 34$ d. and $\angle BDC = 17$ d. 30'. It is re-
quired to find the Distance of the Tower D from
each of the other Towers.

Construction. On AC and BC describe Segments
of Circles capable of containing Angles of 34 d. and
17 d. 30' respectively; then will the Place of their
Intersection, D , be the Point required, as in the last
Case. Pl. VI.
Fig. 5.

The Learner must, by this Time, be able to solve
the common Cases of Plane Trigonometry, and there-
fore we may now save ourselves the Trouble of com-
puting the Numbers, it being sufficient, for the
Future, only to explain the Method of Solution,
leaving the actual Computation for his own Exercise.

Method

Method of Calculation. Conceive the several Radii, &c. to be drawn as in the Figure: Then find the $\angle$ ACB, the Radii GC, and FC, as shewn in the last Case. Also the $\angle$ s FGC and GFC, also CD.

Now the $\angle$ CGB is $\equiv$ twice the $\angle$ CDB by the Construction; also the $\angle$ CGD $\equiv$ twice the $\angle$ CGI (CGF) CG being $\equiv$ GD, and therefore the $\triangle$ CGD Ifofceles. But, by the Figure, $\angle$ BGC + $\angle$ CGD $\equiv$ $\angle$ BGD; and therefore, as the $\triangle$ BGD is Ifofceles, the $\angle$ GBD $\equiv$ $\angle$ GDB, and each $\equiv$ to $\frac{180d. - \angle BGD}{2}$.

Hence, by Trigonometry, BD may be found, by saying, As S. $\angle$ DBG : GD :: S. $\angle$ BGD : BD. In like Manner AD may be found. We have been but short in explaining this, because, if the Learner understands the last Case, what is here said will be sufficient.

Otherwise thus.

152. This Case may be constructed exactly like the second Method given in the last Case, and then the Calculation will be as follows.

Pl. VI.
Fig. 6.

First, in the $\triangle$ ABC are given the three Sides to find the Angle BAC.

Secondly, in the $\triangle$ AEB are given AB and $\angle$ s EAB, EBA, and consequently the $\angle$ AEB, to find AE and EB.

Thirdly, in the $\triangle$ ACE are now given AE, AC, and the contained $\angle$ EAC, ($= \angle$ BAC + $\angle$ EAB,) to find the $\angle$ AEC.

Fourthly, in the $\triangle$ ADE are given AE, $\angle$ AED, (AEC,) and $\angle$ ADE, to find AD.

Fifthly, in the $\triangle$ AED, the $\angle$ s AED and ADE being known, the $\angle$ EAD is also known. But the $\angle$ EAD - $\angle$ EAC $= \angle$ CAD. Hence, in the $\triangle$ CAD are given AC, and $\angle$ s CAD, ADC, to find DC.

Sixthly and lastly, the $\angle$ AEB - $\angle$ AEC $= \angle$ DEB. Hence, in the $\triangle$ DEB are given EB, the $\angle$ s DEB, and EDB, to find BD.

153. *Scholium.* This Case is so much like the last, that perhaps some of our more learned Readers may think we might have contented ourselves with the fourth Case only ; but, as we are writing an Introduction, it seems more excusable to say rather too much than too little.

154. *Case 6.* When the Station falls within the Triangle made by the given Objects.

EXAMPLE. Let A, B, C, represent three Towers, whose Distance from each other is known : And, wanting to find the Distance from another Tower, D, to each of these Towers, I took the Angle $ADC = 116$ d. $\angle BDC = 110$ d. and, to prove the Truth of these Angles, I also took the $\angle ADB = 134$ d. which, together, making 360 d. or a whole Circle, proves them to have been accurately taken.

Pl. VI.
Fig. 7.

Construction. On two of the given Sides describe Segments of Circles capable of containing the given Angles. For Example. On AC a Segment to contain an Angle of 116 d. and on AB one to contain an Angle of 134 d. Then will the Point of Intersection, D, be manifestly the Place required.

Pl. VI.
Fig. 8.

Calculation. Let F and G be the Centers of the two Circles whose Segments are described, and let GH be drawn $\perp$ to AC, and FE $\perp$ to AB, and join F, G ; also conceive the Radii to be drawn as in the Figure.

First, in the $\triangle ABC$ are given the three Sides to find the $\angle CAB$.

Secondly, conceive, in your Mind, the Circle of which the Arc ADC is a Part to be completed, and that Lines be drawn from the Points A and C to any Point in the opposite Segment, so as to form, with the Lines AD, CD, a quadrilateral Figure inscribed in the Circle ; then the $\angle ADC +$ the Angle in the opposite Segment being $= 180$ d. the Angle in the opposite Segment must be $= 180$ d. $- \angle ADC$. And $\frac{1}{2}$ $\angle$ at the Center, viz. $\angle AGH$, $=$ the Angle in that opposite Segment. Hence, in the right-angled $\triangle$

$\triangle$

$\triangle AGH$ are now given $\angle AGH$ and $AH = \frac{1}{2} AC$ to find the Radius AG .

Thirdly, in like Manner, the Angle AFE may be shewn to be $= 180^\circ - \angle ADB$. Hence, in the right-angled $\triangle AEF$ are given $AE = \frac{1}{2} AB$, and $\angle AFE$, to find the Radius AF .

Fourthly, the $\angle GAH$ being $= 90^\circ - \angle AGH$ becomes known; and, in like Manner, the $\angle FAE$ is known. Hence, $\angle FAG$ becomes known, being $= \angle GAH + \angle CAB + \angle FAE$. Hence, in the $\triangle GAF$ are now given the Sides AG, AF , and their contained $\angle GAF$, to find the Angle AGF .

Fifthly, for Reasons already shewn in former Cases, the Line GF , joining the Centers G and F , divides AD in I into two equal Parts, and is perpendicular thereto: Therefore, in the right-angled $\triangle AGI$ we have now given AG and $\angle AGI$ (AGF) to find AI , which doubled gives AD .

Sixthly, hence, in the $\triangle ADC$ are now given two Sides, AC, AD , and $\angle ADC$, to find DC .

Seventhly and lastly, in the $\triangle CDB$ we have now given two Sides, CD, CB , and the $\angle CDB$, to find DB .

Otherwise thus.

155. *Construction.* This may be done in nearly the same Manner as the second Method in Case 4; viz. make the Angle $ABE = \angle ADE$ viz. $= 180^\circ - \angle ADC$; and $\angle BAE = \angle BDE$ viz. $= 180^\circ - \angle BDC$. Then describe a Circle through the three Points A, B, E , and join E, C , by the Line EC ; then will the Point D , where the Line EC intersects the Circle $EADBE$, be the Place of the required Station.

Pl. VI.
Fig. 9.

Demonstration. For, by the Construction, the Angles ABE, ADE , are in the same Segment, $ADBE$, and stand on the same Chord AE , and therefore must be equal, by Art. 146 of our Geometry. Also the Angles BAE, BDE , are equal to each other, being on the same Base, BE , and in the same Segment, $BDAE$.

BDAE. But the Angles ADE, BDE, are Supplements of the Angles ADC, BDC; and the Angles ABE, BAE, were made equal to these Supplements in the Construction; $\therefore$ the Point D is manifestly that required.

Calculation. First, in the $\triangle ABC$ are given the three Sides to find the Angles A and C.

Secondly, in the $\triangle AEB$ are given AB and all the Angles to find AE.

Thirdly, in the $\triangle CAE$ are now given CA, AE, and contained Angle CAE, $= \angle CAB + \angle BAE$, to find the Angle ACE.

Fourthly, In the $\triangle ADC$ are now given AC, and Angles, to find the required Distances AD and DC.

Fifthly and lastly, in the $\triangle CDB$ are now given BC, DC, and their contained Angle DCB, $= \angle ACB - \angle ACD$, to find DB, the last required Distance.

156. *Proposition-6.* Given, the Distance of two Objects, A, B, and the Angles ADB, BDC, BCA, ACD, to find the Distance of the two Stations, D, C, from the Objects, A, B.

Pl. VI.
Fig. 10.

Construction. Draw DC any convenient Length, at Pleasure, and make the $\angle$ s ADB, BDC, BCA, ACD, $=$ the given Angles; and join the Points of Intersection by the Line AB: Then, if AB be made on the Sector $=$ the given Length, DC, DA, &c. may be measured on the Sector.

157. A Sector is generally given in a Case of Pocket-Instruments, and is a very valuable Instrument; but, as its Description and Use would take up more Room than we can spare in this Place, we must defer its Description, &c. till some other Opportunity: And, therefore, for the Sake of such as are not acquainted with the Use of a Sector, it is necessary to give another Construction, viz.

Make dc $=$ any Number, at Pleasure, and make the Angles bdc , adc , acd , bca , respectively, $=$ the given Angles BDC, ADC, ACD, BCA, and join a, b ; then it is plain this Figure must be similar to that re-

M

quired:

quired: Therefore, having drawn $AB =$ the given Distance, make the Angle $ABC = \angle abc$, and $\angle BAC = \angle bac$; also the $\angle$ s BAD , ABD , respectively, $=$ the Angles bad , abd ; then will the Points of Interfection, D , C , be those required. Join D , C .

Calculation. In the $\triangle adc$ are given $dc =$ the assumed Number, the Angle $adc = adb + bdc$, and Angle acd , and consequently the remaining Angle dac , to find ad , ac .

In like Manner, in the $\triangle bcd$, are given the three Angles and the Side dc to find db , bc .

Now, in the $\triangle adb$ are given the two Sides ad , bd , and contained Angle adb , to find ab . Hence, by the Nature of similar Figures, we have, As $ab : AB :: dc : DC$, and $:: ad : AD$, $:: ac : AC$, $:: bd : BD$, and $:: bc : BC$.

Pl. VI. 158. *Proposition 7.* Given, the Distances of 3
Fig. 11. Objects, A , B , C , from each other, the $\angle$ s ADC , CDE , CED , and CEB , to find the Sides AD , DC , DE , EC , and EB .

Pl. VI. *Construction.* Assume any Line, de , at Pleasure;
Fig. 12. make the Angle $cde = \angle CDE$, and Angle $ced = \angle CED$; also the Angle $cda = \angle CDA$, and $\angle ceb = \angle CEB$. Produce ad and be to intersect each other at f , and join cf .

Then, it is manifest, the Figures $cdfe$, $CDFE$, are similar. Therefore, on AC describe a Segment of a Circle capable of containing an $\angle AFC = \angle asc$; and on CB a Segment capable of containing an Angle $CFB = \angle cfb$. From the Point of Interfection, F , draw FA , FB , FC . Make the Angle $FCI = \angle fcd$, and $\angle FCE = \angle fce$; which completes the Construction.

Calculation. Assume $de =$ any Number; then, in the $\triangle def$ are given de , $\angle fde = 180^\circ -$ the Sum of the Angles adc , cde , and $\angle def = 180^\circ -$ the Sum of the $\angle$ s bec , ced , and consequently the remaining $\angle dfe$, to find df and fe . Also, in the $\triangle dce$, the Angles and Side de are given, to find the Sides dc and

and ce . Hence, in the $\triangle dcf$ are now given dc , df , and contained $\angle cdf$, to find $\angle dfc$. Hence, also, $\angle cfe$ becomes known, being $= \angle dfe - \angle dfc$. We have now given AC and Angle AFC , also BC and Angle BFC , which reduces the Problem now to the same as Prop. 5, Case 4, by which may be found FA , FC , FB : Then, by similar Triangles, As $fc : cd :: FC : CD$; and as $fc : ce :: FC : CE$.

Now, in the $\triangle ACD$ are given AC , DC , and $\angle ADC$, to find AD .

Lastly, in the $\triangle CBE$ are given CB , CE , and Angle BEC , to find BE .

159. *Scholium.* This Solution fails when AD is parallel to BC ; or, which is the same, when the Angle $ADE + BED$ is $= 180^\circ$. For, in such Case, from the Nature of Parallels, AD and BE , produced, could never meet. For which Reason, it may be proper to give another Method of Solution.

Construction. Having described the Segments ADC , BEC , to contain the given Angles ADC and BEC , respectively, draw the Line CF to cut off a Segment $=$ Angle CDF , and the Right-Line CG to cut off a Segment $=$ Angle CEG ; then, it is manifest, the Points G , F , must be in the Right-Line DE : Therefore, join G , F , and produce the Line each Way till it intersects the Segments, then will the Points of Intersection, D and E , be the Stations required.

Pl. VI.
Fig. 13.

Calculation. Conceive the Radii and Perpendiculars drawn, as in the Figure is shewn by the dotted Lines; then, by Trigonometry, in the right-angled $\triangle CIH$, we have, As $S. \angle IHC (= \text{Angle } ADC)$: $IC (\frac{1}{2} AC) :: R : HC$.

Secondly, in the right-angled $\triangle CKH$, As $R :: HC :: S. \angle KHC (= \text{Angle } CDF) : CK$. Then $2 CK = CF$.

Thirdly, in like Manner, by the other Segment, may be found CG .

Fourthly, the Angle $ICH = 90^\circ - \angle IHC$; and $HCK = 90^\circ - \angle KHC$; then Angle $ICH + \angle HCK = \angle ACF$. In like Manner, the Angle BCG becomes known, being equal to the Angle $MCL + \angle LCN$. But the Angle $ACF + \angle BCG - \angle ACB = \angle GCF$.

Fifthly, hence, in the $\triangle GCF$ are now given GC , FC , and the contained Angle GCF , to find the other Angles. Hence, the Angles CGD , CFE , become known: And, the Trapezium $ACFDA$ being in a Circle, the Angle $DAC + \angle DFC = 180^\circ$, and consequently Angle $DAC = 180^\circ - \angle DFC$. In like Manner the Angle EBC becomes known.

Sixthly, hence, in the $\triangle DAC$ we have now given AC , and Angles, to find DC and DA .

Seventhly, in like Manner, in the $\triangle EBC$ are given BC , and the Angles, to find EC and EB .

Eighthly and lastly, in the $\triangle DCE$ are now given DC , EC , and all the Angles, to find DE ; and so the Problem will be completely solved.

160. *Scholium.* From this Construction it appears, that, when the Points G and F coincide, the Proposition is indeterminate; as, in such Case, the Line DE may be drawn in any Direction.

On my communicating this Problem to a Friend, he brought me a Solution nearly the same, in Substance, as the second Method here given.

161. If Room would have permitted in the Plate, I intended to have given other Methods of Solution. Would Room admit, I might also have added several other Problems; but these are sufficient to shew the

* If the Truth of this Equation should not be sufficiently plain, it may be shewn thus:

$$\text{For the } \angle ACG + \angle GCF = \angle ACF$$

$$\text{And } \angle BCF + \angle GCF = \angle BCG$$

$$\text{Sum } \angle ACG + \angle BCF + 2\angle GCF = \angle ACF + \angle BCG:$$

$$\text{But } \angle ACG + \angle BCF + \angle GCF = \angle ACB.$$

By Subtraction, the $\angle GCF = \angle ACF + \angle BCG - \angle ACB$, the same as above.

the great Usefulness of Trigonometry in taking Heights and Distances of Places, and in surveying of Counties, Sea-Coasts, and Harbours; and, if the young Student will make himself Master of these, it may be supposed he will not meet with any Difficulties, in real Practice, which he cannot readily overcome, as very few of our Surveyors even take the Pains to understand thus much. However, we may, perhaps, hereafter, take another Opportunity to add some more Problems, either in our intended Treatise of Surveying, or in a Treatise to be entitled *Trigonometrical Essays*. In which, if Time, Health, and Encouragement permit, I propose to shew, first, A Method of constructing Tables of natural and artificial Sines and Tangents, &c. with the Construction of Scales of Sines, Tangents, &c. as hinted at in Art. 26 of this Essay. Secondly, The Construction and Use of the Sector, &c. Thirdly, The Method of solving Triangles, independent of Tables or Instruments. Fourthly, The algebraical and geometrical Solutions to a Variety of Questions; with the Method of deducing geometrical Constructions from algebraical Solutions, &c.

162. Agreeable to our Promise, in Page 12, we shall conclude this Essay with the following useful Table.

A TABLE of Natural-Sines, Tangents, and Secants.

RADIUS = 1.

D	M	Sine.	Cofine.	Tang.	Co-Tan.	Sec.	Co-Sec.	M	D
0	0	.0000	1.000	.0000	Infinite.	1.000	Infinite.	0	90
	1	.0003	1.000	.0003	3438.	1.000	3438.	59	
	10	.0029	1.000	.0029	343.8	1.000	343.8	50	
	20	.0058	1.000	.0058	171.9	1.000	171.9	40	
	30	.0087	1.000	.0087	114.6	1.000	114.6	30	
	40	.0116	.9999	.0116	85.94	1.000	85.95	20	
	50	.0145	.9999	.0145	68.75	1.000	68.76	10	
1	0	.0174	.9998	.0174	57.29	1.000	57.30	0	89
	10	.0203	.9998	.0203	49.10	1.000	49.11	50	
	20	.0233	.9997	.0233	42.96	1.000	42.97	40	
	30	.0262	.9996	.0262	38.18	1.000	38.20	30	
	40	.0291	.9995	.0291	34.36	1.000	34.38	20	
	50	.0320	.9994	.0320	31.24	1.000	31.26	10	
2	0	.0349	.9994	.0349	28.64	1.000	28.65	0	88
	10	.0378	.9993	.0378	26.43	1.000	26.45	50	
	20	.0407	.9992	.0407	24.54	1.000	24.56	40	
	30	.0436	.9990	.0437	22.90	1.000	22.92	30	
	40	.0465	.9990	.0466	21.47	1.001	21.49	20	
	50	.0494	.9988	.0495	20.21	1.001	20.23	10	
3	0	.0523	.9986	.0524	19.08	1.001	19.11	0	87
	10	.0552	.9985	.0553	18.07	1.001	18.10	50	
	20	.0581	.9983	.0582	17.17	1.002	17.20	40	
	30	.0610	.9981	.0612	16.35	1.002	16.38	30	
	40	.0640	.9980	.0640	15.60	1.002	15.63	20	
	50	.0668	.9980	.0670	14.92	1.002	14.96	10	
4	0	.0698	.9976	.0699	14.30	1.002	14.34	0	86
	10	.0726	.9973	.0728	13.73	1.003	13.76	50	
	20	.0756	.9971	.0757	13.20	1.003	13.23	40	
	30	.0784	.9969	.0787	12.71	1.003	12.74	30	
	40	.0813	.9967	.0816	12.25	1.003	12.29	20	
	50	.0843	.9964	.0846	11.26	1.003	11.87	10	
5	0	.0871	.9962	.0875	11.43	1.004	11.47	0	85
	10	.0901	.9960	.0904	11.06	1.004	11.10	50	
	20	.0929	.9957	.0933	10.71	1.004	10.76	40	
	30	.0958	.9953	.0963	10.39	1.004	10.43	30	
	40	.0990	.9951	.0992	10.08	1.005	10.13	20	
	50	.1016	.9948	.1021	9.788	1.005	9.839	10	
5	60	.1045	.9945	.1051	9.514	1.006	9.567	0	84
		Cofine.	Sine.	Co-Ta.	Tangent.	Co-S.	Secant.		

Natural-Sines, Tangents, and Secants.

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D.	M.	Sines.	Cofines	Tang.	Co-Ta.	Secants	Co-Se.	M.	D.
6	0	.1045	.9945	.1051	9.514	1.006	9.567	0	84
	10	.1074	.9942	.1080	9.255	1.006	9.309	50	
	20	.1103	.9939	.1109	9.009	1.006	9.065	40	
	30	.1132	.9936	.1139	8.777	1.006	8.834	30	
	40	.1161	.9932	.1169	8.556	1.007	8.613	20	
	50	.1190	.9929	.1198	8.345	1.007	8.404	10	
7	0	.1219	.9925	.1228	8.144	1.007	8.205	0	85
	10	.1247	.9922	.1257	7.953	1.008	8.016	50	
	20	.1276	.9918	.1287	7.770	1.008	7.834	40	
	30	.1305	.9914	.1316	7.596	1.009	7.661	30	
	40	.1334	.9910	.1346	7.428	1.009	7.495	20	
	50	.1363	.9907	.1376	7.269	1.009	7.337	10	
8	0	.1392	.9902	.1405	7.115	1.009	7.185	0	82
	10	.1420	.9899	.1436	6.968	1.010	7.040	50	
	20	.1449	.9894	.1464	6.826	1.010	6.900	40	
	30	.1478	.9890	.1494	6.691	1.011	6.765	30	
	40	.1507	.9886	.1524	6.560	1.011	6.636	20	
	50	.1536	.9881	.1554	6.435	1.012	6.512	10	
9	0	.1564	.9877	.1583	6.313	1.012	6.392	0	81
	10	.1593	.9872	.1614	6.197	1.013	6.277	50	
	20	.1622	.9868	.1643	6.084	1.013	6.166	40	
	30	.1650	.9862	.1673	5.976	1.014	6.059	30	
	40	.1680	.9859	.1703	5.871	1.014	5.955	20	
	50	.1708	.9853	.1733	5.769	1.015	5.855	10	
10	0	.1736	.9848	.1763	5.671	1.015	5.759	0	80
	10	.1765	.9843	.1793	5.576	1.016	5.665	50	
	20	.1793	.9838	.1823	5.484	1.016	5.575	40	
	30	.1822	.9832	.1853	5.395	1.017	5.487	30	
	40	.1851	.9827	.1883	5.309	1.017	5.402	20	
	50	.1880	.9822	.1914	5.226	1.018	5.320	10	
11	0	.1908	.9816	.1944	5.144	1.018	5.240	0	79
	10	.1936	.9811	.1974	5.065	1.019	5.163	50	
	20	.1965	.9805	.2004	4.989	1.019	5.089	40	
	30	.1994	.9799	.2035	4.915	1.020	5.016	30	
	40	.2022	.9793	.2064	4.843	1.021	4.945	20	
	50	.2051	.9787	.2095	4.773	1.022	4.876	10	
11	60	.2079	.9781	.2125	4.704	1.022	4.809	0	78
		Cofine	Sine.	Co Ta.	Tang.	Co-Se.	Secants		

Natural-Sines, Tangents, and Secants.

D.	M.	Sines.	Cofines	Tang.	Co-Ta.	Secants	Co-sec.	M.	D.
12	0	.2079	.9781	.2125	4.704	1.022	4.809	0	78
	10	.2107	.9775	.2155	4.638	1.023	4.745	50	
	20	.2136	.9769	.2186	4.574	1.023	4.682	40	
	30	.2164	.9763	.2217	4.511	1.024	4.620	30	
	40	.2193	.9757	.2247	4.449	1.025	4.560	20	
	50	.2221	.9750	.2278	4.390	1.025	4.502	10	
13	0	.2249	.9744	.2309	4.331	1.026	4.445	0	77
	10	.2278	.9737	.2339	4.274	1.027	4.390	50	
	20	.2306	.9730	.2370	4.219	1.027	4.336	40	
	30	.2334	.9724	.2401	4.165	1.028	4.283	30	
	40	.2362	.9717	.2431	4.112	1.029	4.232	20	
	50	.2391	.9710	.2462	4.061	1.029	4.182	10	
14	0	.2419	.9703	.2493	4.011	1.031	4.133	0	76
	10	.2447	.9696	.2524	3.961	1.031	4.086	50	
	20	.2476	.9688	.2555	3.913	1.032	4.039	40	
	30	.2504	.9681	.2586	3.866	1.033	3.994	30	
	40	.2532	.9674	.2617	3.820	1.033	3.949	20	
	50	.2560	.9667	.2648	3.776	1.034	3.906	10	
15	0	.2588	.9659	.2679	3.732	1.035	3.864	0	75
	10	.2616	.9651	.2710	3.689	1.036	3.822	50	
	20	.2644	.9644	.2742	3.647	1.036	3.781	40	
	30	.2672	.9636	.2773	3.605	1.037	3.742	30	
	40	.2700	.9628	.2804	3.565	1.038	3.703	20	
	50	.2728	.9620	.2836	3.526	1.039	3.665	10	
16	0	.2756	.9613	.2867	3.487	1.040	3.628	0	74
	10	.2784	.9604	.2899	3.449	1.041	3.591	50	
	20	.2812	.9596	.2931	3.412	1.042	3.556	40	
	30	.2840	.9588	.2962	3.376	1.043	3.521	30	
	40	.2868	.9580	.2994	3.340	1.043	3.487	20	
	50	.2896	.9571	.3025	3.305	1.044	3.453	10	
17	0	.2924	.9563	.3057	3.270	1.046	3.420	0	73
	10	.2951	.9554	.3089	3.237	1.046	3.388	50	
	20	.2979	.9546	.3121	3.204	1.047	3.356	40	
	30	.3007	.9537	.3153	3.171	1.048	3.325	30	
	40	.3035	.9528	.3185	3.139	1.049	3.295	20	
	50	.3062	.9519	.3217	3.108	1.050	3.265	10	
17	60	.3090	.9510	.3249	3.077	1.051	3.236	0	72
		Cofine.	Sines.	Co-Ta.	Tang.	Co-sec.	Secants		

D.	M.	Sines.	Cofines	Tang.	Co-Ta.	Secants	Co-Se.	M.	D.
18	0	.3090	.9510	.3249	3.077	1.051	3.236	0	72
	10	.3118	.9501	.3281	3.047	1.052	3.207	50	
	20	.3145	.9492	.3314	3.017	1.053	3.179	40	
	30	.3173	.9483	.3346	2.988	1.054	3.151	30	
	40	.3200	.9474	.3378	2.960	1.055	3.124	20	
	50	.3228	.9464	.3410	2.932	1.056	3.098	10	
19	0	.3256	.9455	.3443	2.904	1.058	3.071	0	71
	10	.3283	.9446	.3476	2.877	1.058	3.045	50	
	20	.3310	.9436	.3508	2.850	1.059	3.020	40	
	30	.3338	.9426	.3541	2.824	1.061	2.996	30	
	40	.3365	.9417	.3574	2.798	1.062	2.971	20	
	50	.3393	.9407	.3607	2.772	1.063	2.947	10	
20	0	.3420	.9397	.3640	2.747	1.064	2.924	0	70
	10	.3447	.9387	.3673	2.723	1.065	2.901	50	
	20	.3475	.9377	.3706	2.698	1.066	2.878	40	
	30	.3502	.9367	.3739	2.675	1.069	2.883	30	
	40	.3529	.9356	.3772	2.651	1.069	2.833	20	
	50	.3556	.9346	.3805	2.628	1.070	2.812	10	
21	0	.3584	.9336	.3839	2.605	1.071	2.790	0	69
	10	.3611	.9325	.3872	2.583	1.072	2.769	50	
	20	.3638	.9315	.3905	2.560	1.073	2.749	40	
	30	.3665	.9304	.3939	2.539	1.075	2.728	30	
	40	.3692	.9293	.3973	2.517	1.076	2.708	20	
	50	.3719	.9283	.4006	2.496	1.077	2.689	10	
22	0	.3746	.9272	.4040	2.475	1.078	2.669	0	68
	10	.3773	.9261	.4074	2.454	1.080	2.650	50	
	20	.3800	.9250	.4108	2.434	1.081	2.632	40	
	30	.3827	.9239	.4142	2.414	1.082	2.613	30	
	40	.3854	.9227	.4176	2.394	1.084	2.595	20	
	50	.3880	.9216	.4210	2.375	1.085	2.577	10	
23	0	.3907	.9205	.4245	2.356	1.086	2.559	0	67
	10	.3934	.9194	.4279	2.337	1.088	2.542	50	
	20	.3961	.9182	.4313	2.318	1.089	2.525	40	
	30	.3987	.9171	.4348	2.300	1.090	2.508	30	
	40	.4014	.9159	.4383	2.282	1.092	2.491	20	
	50	.4041	.9147	.4417	2.264	1.093	2.475	10	
24	60	.4067	.9135	.4452	2.246	1.095	2.458	0	66
		Cofine.	Sine.	Co-Ta.	Tang.	Co-Se.	Secant.		

Natural-Sines, Tangents, and Secants.

D.	M.	Sine.	Cofines	Tang.	Co-Ta.	Secants	Co-Se.	M.	D.
24	0	.4067	.9135	.4452	2.246	1.095	2.458	0	66
	10	.4094	.9123	.4487	2.228	1.096	2.443	50	
	20	.4120	.9112	.4522	2.213	1.097	2.427	40	
	30	.4147	.9100	.4557	2.194	1.099	2.411	30	
	40	.4173	.9087	.4592	2.177	1.100	2.396	20	
	50	.4200	.9075	.4628	2.161	1.102	2.381	10	
25	0	.4226	.9063	.4631	2.144	1.103	2.366	0	65
	10	.4252	.9051	.4698	2.128	1.105	2.351	50	
	20	.4279	.9038	.4734	2.112	1.106	2.337	40	
	30	.4305	.9026	.4770	2.096	1.108	2.323	30	
	40	.4331	.9013	.4805	2.081	1.109	2.309	20	
	50	.4357	.9001	.4841	2.065	1.111	2.295	10	
26	0	.4384	.8988	.4877	2.050	1.113	2.281	0	64
	10	.4410	.8975	.4913	2.035	1.114	2.268	50	
	20	.4436	.8962	.4950	2.020	1.116	2.254	40	
	30	.4462	.8949	.4986	2.006	1.117	2.241	30	
	40	.4488	.8936	.5022	1.991	1.119	2.228	20	
	50	.4514	.8923	.5058	1.977	1.120	2.215	10	
27	0	.4540	.8910	.5095	1.963	1.122	2.203	0	63
	10	.4565	.8897	.5132	1.948	1.124	2.190	50	
	20	.4592	.8883	.5169	1.935	1.126	2.178	40	
	30	.4617	.8870	.5200	1.921	1.127	2.166	30	
	40	.4643	.8856	.5243	1.907	1.129	2.154	20	
	50	.4669	.8843	.5280	1.894	1.131	2.142	10	
28	0	.4695	.8829	.5317	1.881	1.132	2.130	0	62
	10	.4720	.8816	.5354	1.867	1.134	2.118	50	
	20	.4746	.8802	.5392	1.855	1.136	2.107	40	
	30	.4771	.8788	.5429	1.842	1.138	2.096	30	
	40	.4797	.8774	.5467	1.829	1.139	2.084	20	
	50	.4823	.8760	.5505	1.816	1.141	2.073	10	
29	0	.4848	.8746	.5543	1.804	1.143	2.063	0	61
	10	.4873	.8732	.5581	1.792	1.145	2.052	50	
	20	.4899	.8718	.5619	1.779	1.147	2.041	40	
	30	.4924	.8703	.5658	1.767	1.149	2.030	30	
	40	.4949	.8689	.5696	1.755	1.151	2.020	20	
	50	.4975	.8675	.5735	1.743	1.153	2.010	10	
29	50	.5000	.8660	.5773	1.732	1.155	2.000	0	60
		Cofine.	Sine.	Co-Ta	Tang.	Co-Se.	Secant.		

D.	M.	Sines.	Cofines	Tang.	Co-Ta.	Secants	Co-Se.	M.	
30	0	.5000	.8660	.5773	1.732	1.155	2.000	0	60
	10	.5025	.8645	.5812	1.720	1.156	1.990	50	
	20	.5050	.8631	.5851	1.709	1.158	1.980	40	
	30	.5075	.8616	.5890	1.698	1.160	1.970	30	
	40	.5100	.8601	.5930	1.686	1.162	1.961	20	
	50	.5125	.8586	.5969	1.675	1.165	1.951	10	
31	0	.5150	.8572	.6008	1.664	1.166	1.942	0	59
	10	.5175	.8556	.6048	1.653	1.168	1.932	50	
	20	.5200	.8541	.6088	1.642	1.171	1.923	40	
	30	.5225	.8526	.6128	1.632	1.173	1.914	30	
	40	.5249	.8511	.6168	1.621	1.175	1.905	20	
	50	.5274	.8496	.6208	1.611	1.177	1.896	10	
32	0	.5299	.8480	.6248	1.600	1.179	1.887	0	58
	10	.5324	.8465	.6289	1.590	1.181	1.878	50	
	20	.5348	.8449	.6330	1.580	1.183	1.870	40	
	30	.5373	.8434	.6371	1.570	1.186	1.861	30	
	40	.5397	.8418	.6412	1.559	1.188	1.853	20	
	50	.5422	.8402	.6453	1.550	1.190	1.844	10	
33	0	.5446	.8387	.6494	1.540	1.192	1.836	0	57
	10	.5471	.8371	.6535	1.530	1.195	1.828	50	
	20	.5495	.8355	.6577	1.520	1.197	1.820	40	
	30	.5519	.8339	.6619	1.511	1.199	1.812	30	
	40	.5544	.8323	.6661	1.501	1.201	1.804	20	
	50	.5568	.8306	.6703	1.492	1.203	1.796	10	
34	0	.5592	.8290	.6745	1.482	1.206	1.788	0	56
	10	.5616	.8274	.6787	1.473	1.208	1.780	50	
	20	.5640	.8258	.6830	1.464	1.211	1.773	40	
	30	.5664	.8241	.6873	1.455	1.213	1.765	30	
	40	.5688	.8225	.6916	1.446	1.216	1.758	20	
	50	.5712	.8208	.6958	1.437	1.218	1.751	10	
35	0	.5736	.8191	.7002	1.428	1.220	1.743	0	55
	10	.5759	.8175	.7045	1.419	1.223	1.736	50	
	20	.5783	.8158	.7089	1.411	1.226	1.729	40	
	30	.5807	.8141	.7133	1.402	1.228	1.722	30	
	40	.5831	.8124	.7177	1.393	1.231	1.715	20	
	50	.5854	.8107	.7221	1.385	1.233	1.708	10	
35	60	.5878	.8090	.7265	1.376	1.236	1.701	0	54
		Cofine.	Sine.	Co Ta.	Tang.	Co-Se.	Secant.		

Natural-Sines, Tangents, and Secants.

D.	M.	Sines.	Cofines	Tang.	Co-Ta.	Secants	Co-sec.	M.	D.
36	0	.5878	.8090	.7265	1.376	1.236	1.701	0	54
	10	.5901	.8073	.7310	1.368	1.239	1.694	50	
	20	.5925	.8056	.7355	1.359	1.241	1.688	40	
	30	.5948	.8038	.7400	1.351	1.244	1.681	30	
	40	.5971	.8021	.7445	1.343	1.247	1.674	20	
	50	.5995	.8004	.7490	1.335	1.249	1.668	10	
37	0	.6018	.7990	.7535	1.327	1.252	1.662	0	53
	10	.6041	.7969	.7581	1.319	1.255	1.655	50	
	20	.6064	.7951	.7627	1.311	1.258	1.649	40	
	30	.6088	.7933	.7673	1.303	1.260	1.643	30	
	40	.6111	.7916	.7719	1.295	1.263	1.636	20	
	50	.6134	.7898	.7766	1.287	1.266	1.630	10	
38	0	.6157	.7880	.7813	1.280	1.269	1.624	0	52
	10	.6179	.7862	.7860	1.272	1.272	1.618	50	
	20	.6202	.7844	.7907	1.265	1.275	1.612	40	
	30	.6225	.7826	.7954	1.257	1.278	1.606	30	
	40	.6248	.7808	.8002	1.249	1.281	1.600	20	
	50	.6270	.7790	.8050	1.242	1.284	1.595	10	
39	0	.6293	.7771	.8098	1.235	1.287	1.589	0	51
	10	.6316	.7753	.8146	1.227	1.290	1.583	50	
	20	.6338	.7735	.8195	1.220	1.293	1.578	40	
	30	.6361	.7716	.8243	1.213	1.296	1.572	30	
	40	.6383	.7698	.8292	1.206	1.299	1.567	20	
	50	.6405	.7679	.8341	1.199	1.302	1.561	10	
40	0	.6428	.7660	.8391	1.192	1.305	1.556	0	50
	10	.6450	.7642	.8441	1.185	1.309	1.550	50	
	20	.6472	.7623	.8491	1.178	1.312	1.545	40	
	30	.6494	.7604	.8541	1.171	1.315	1.540	30	
	40	.6516	.7585	.8591	1.164	1.318	1.534	20	
	50	.6539	.7566	.8642	1.157	1.322	1.529	10	
41	0	.6560	.7547	.8693	1.150	1.325	1.524	0	49
	10	.6582	.7528	.8744	1.143	1.328	1.519	50	
	20	.6604	.7509	.8795	1.137	1.332	1.514	40	
	30	.6626	.7489	.8847	1.130	1.335	1.509	30	
	40	.6648	.7470	.8899	1.124	1.339	1.504	20	
	50	.6670	.7670	.8951	1.117	1.342	1.499	10	
41	60	.6691	.7431	.9004	1.111	1.346	1.494	0	48
		Cofine.	Sine.	Co-Ta.	Tang.	Co-sec.	Secant.		

D.	M.	Sines.	Cofines	Tang.	Co-Ta.	Secants	Co-Se.	M.	D.
42	0	.6691	.7431	.9004	1.111	1.346	1.494	0	48
	10	.6713	.7412	.9057	1.104	1.349	1.490	50	
	20	.6734	.7392	.9020	1.098	1.353	1.485	40	
	30	.6756	.7373	.9163	1.091	1.356	1.480	30	
	40	.6773	.7353	.9217	1.085	1.360	1.475	20	
	50	.6799	.7333	.9271	1.079	1.364	1.471	10	
43	0	.6820	.7313	.9325	1.072	1.367	1.466	0	47
	10	.6841	.7294	.9380	1.066	1.371	1.462	50	
	20	.6862	.7273	.9434	1.060	1.375	1.457	40	
	30	.6883	.7254	.9490	1.054	1.379	1.453	30	
	40	.6905	.7234	.9545	1.048	1.382	1.448	20	
	50	.6926	.7213	.9601	1.041	1.386	1.444	10	
44	0	.6946	.7193	.9657	1.035	1.390	1.439	0	46
	10	.6967	.7173	.9713	1.029	1.394	1.435	50	
	20	.6988	.7153	.9770	1.023	1.398	1.431	40	
	30	.7009	.7132	.9827	1.018	1.402	1.427	30	
	40	.7030	.7112	.9884	1.012	1.406	1.422	20	
	50	.7050	.7092	.9942	1.006	1.410	1.418	10	
44	60	.7071	.7071	1.000	1.000	1.414	1.414	0	45
		Cofine.	Sine.	Co-Ta.	Tang.	Co-Se.	Secant.		

THE END OF THE ELEMENTS OF TRIGONOMETRY.

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I N D E X

TO THE

WHOLE VOLUME.

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DE r. *BC* $\times$ *DC.* P. 80. l. 5. for *FG* r. *FH.* P. 81. l. 3. for *other* *Leg* r.
other Ruler. P. 83. l. 2. for $\triangle EBG$ r. $\angle EBG$; *ibid.* l. 18. for $\angle EBG$ r.
 $\angle D$; and *ibid.* l. 10. *b.* for *HFK* r. *HKE.* P. 86. l. 8. for *it* r. *BF.* P. 90.
l. 16. for $\angle KBA = \angle DEF$ r. $\angle BKA = \angle DEG.$ P. 91. l. 9. *b.* for
 $\perp$ to *it* the *Diameter AC* r. $\perp$ to *it* the *Diameter BD.* P. 93. l. 15. for *GBA*
r. *GBE.* P. 111. l. 3. for $DE^2 = 225$ r. $DC^2 = 225$; and for $DC^2 = 325$ r.
 $DF^2 = 325$; and for $28)125$ r. $28)225.$ P. 131. *delo also* for *7632407* r.
7629.4317.

N. B. b signifies to count from the Bottom of the Page.

DIRECTIONS TO THE BINDER.

The several Parts are to be bound in the following Order.

1. The general Title and Preface.
2. The *Essay on Logarithms.*
3. ————— *Geometry.*
4. ————— *Trigonometry.*

The Plates which belong to the *Essay on Geometry* are to be thus placed.

Plate I. at Page 18: II. at P. 34: III. at P. 42: IV. at P. 66:
V. at P. 84: VI. at P. 92: VII. at P. 106: And VIII. at
P. 120.

The six Plates of *Trigonometry* may be placed at the End.



GENERAL HEADS

OF A

Course of Lectures

In Experimental Philosophy.

By BENJAMIN DONN.

I. PROPERTIES OF MATTER; *viz.* its Divisibility, Laws of Motion, &c. — Attraction of Cohesion; or that Quality by which the Particles of Matter adhere to each other and form larger Bodies, &c. *One Lecture.*

II. OPTICS; or the Science of Vision, &c. *Two Lectures.*

III. ELECTRICITY. The electrical Attraction and Repulsion shewn, in a Variety of Experiments. — Its Effects on Animals, &c. — Of the electrical Shock, Lightning, &c. — To preserve Ships, &c. *Two Lectures.*

IV. MAGNETISM and GRAVITY; *viz.* of the Making and Use of Magnets. — Of the Variation of the Compass. — Of Gravity. — Laws of falling Bodies. — Pendulums. — Expansion of Metals. — To discover *true* and *false* Balances. *One Lecture.*

V. MECHANICS; shewing an easy Method of computing the Powers of all Kinds of Machines, though ever so much compounded. *One Lecture.*

VI. HYDROSTATICS; or the Equilibrium and Pressure of Fluids in general. *One Lecture.*

VII. PNEUMATICS; or the Properties of Air, shewn by a Variety of Experiments on the Air and other Pumps, &c. *Two Lectures.*

VIII. GEOGRAPHY and ASTRONOMY. The Figure of the Earth. — Of Latitudes and Longitudes of Places, &c. — Principal Phenomena of the Heavens; illustrating and proving the Motions

General Heads of a Course of Lectures.

tions of the Earth, Planets, and Comets. — Cause of the Seasons; Eclipses, &c. *Four Lectures.*

The Experiments will be exhibited on a PROPER APPARATUS, and the whole explained in so clear a Manner, that, by this Course, Ladies and Gentlemen, without any previous Knowledge of these Subjects, may readily understand the principal Phenomena of Nature, which have been discovered by the most learned Philosophers, in several Ages.

ADMITTANCE to the *whole Course* ONE GUINEA.

At *Christmas* or *Midsummer* Vacation Mr. DONN will go to any Town (not more than thirty Miles distant from *Taunton*) for twenty Subscribers certain; to any Place, within fifty Miles, for thirty Subscribers: Greater Distance in Proportion. He desires about three Months previous Notice.

Sometimes the following ADDITIONAL LECTURES are read, at *One Shilling* and *Six-Pence* each Lecture, *viz.*

I. A Lecture on OPTICS, containing some prismatic Experiments, with an Exhibition of Objects by the solar Microscope.

II. *One* on GEOMETRY, in which the principal Propositions are mechanically proved.

III. *One* on GEOGRAPHY, containing the Construction of Maps, &c.

When the above Course is read in the Academy, it is generally comprised in about twenty-six Lectures.

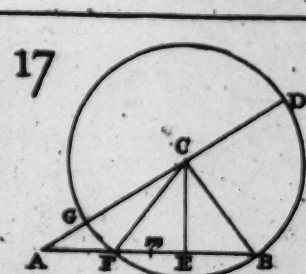
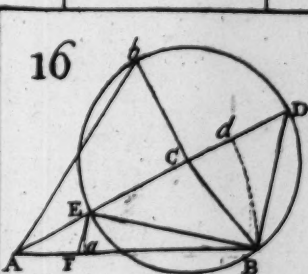
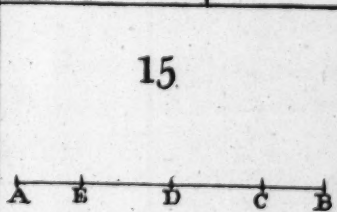
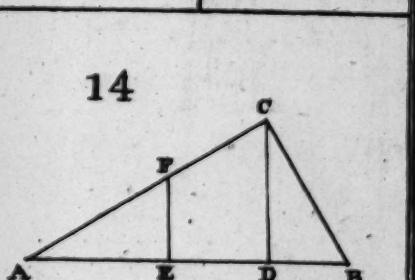
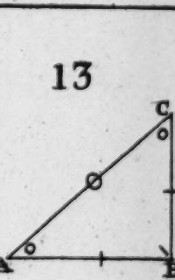
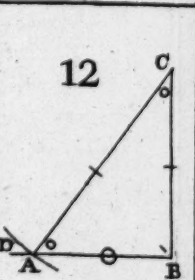
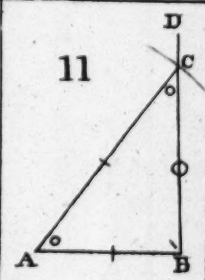
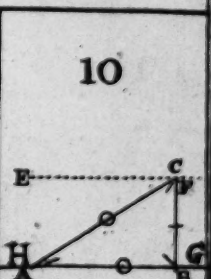
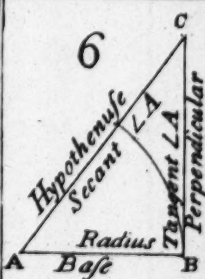
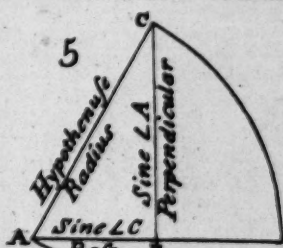
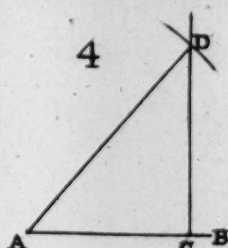
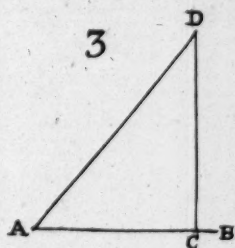
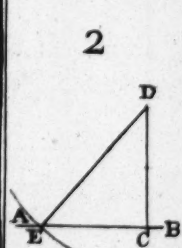


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Trigonometry

1

Plate II

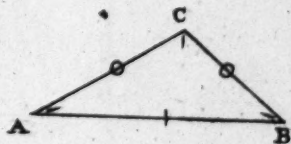




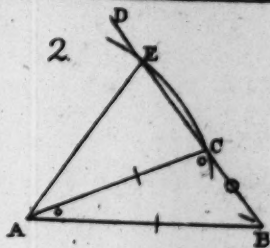
Trigonometry

Plate III

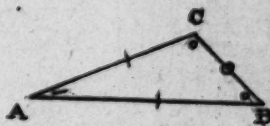
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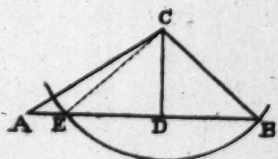
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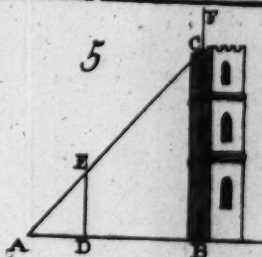
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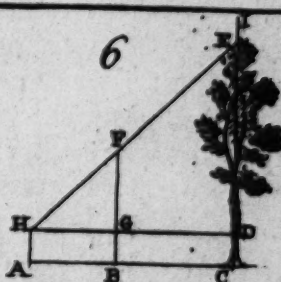
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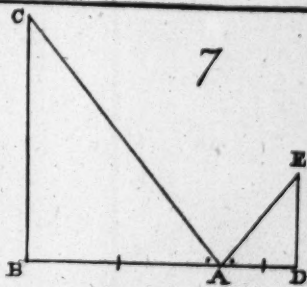
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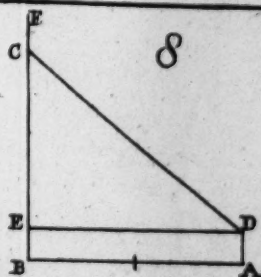
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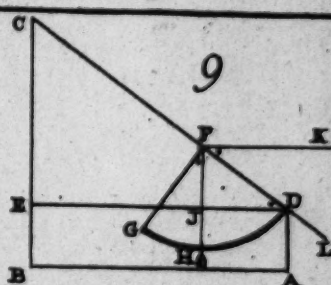
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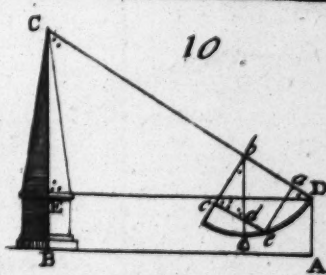
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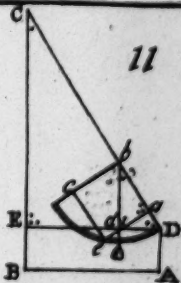
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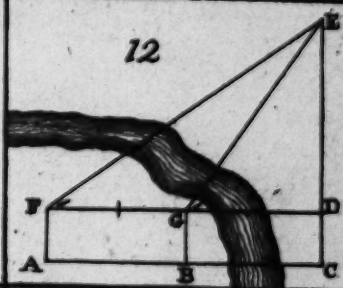
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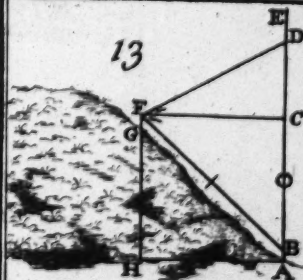
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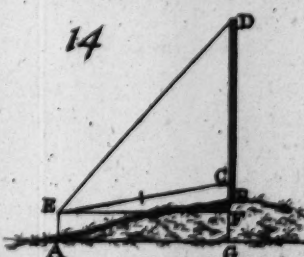
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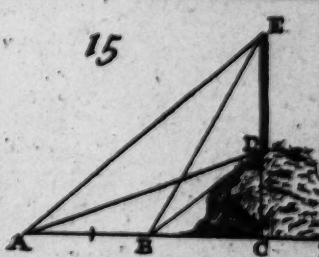
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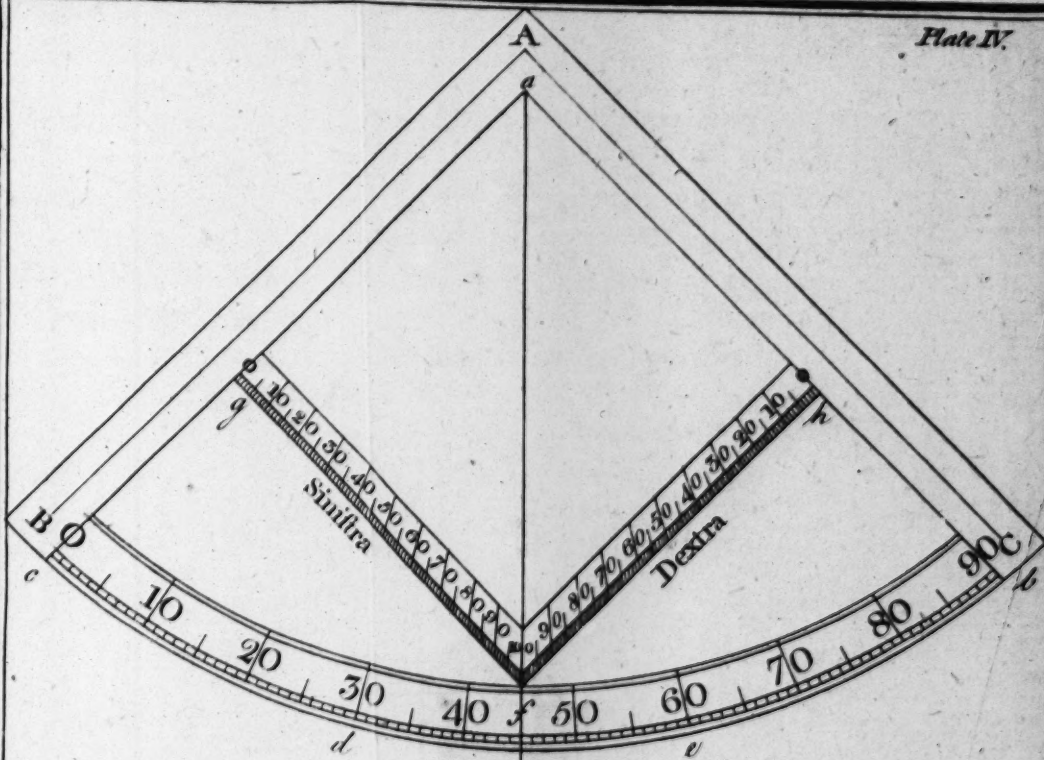
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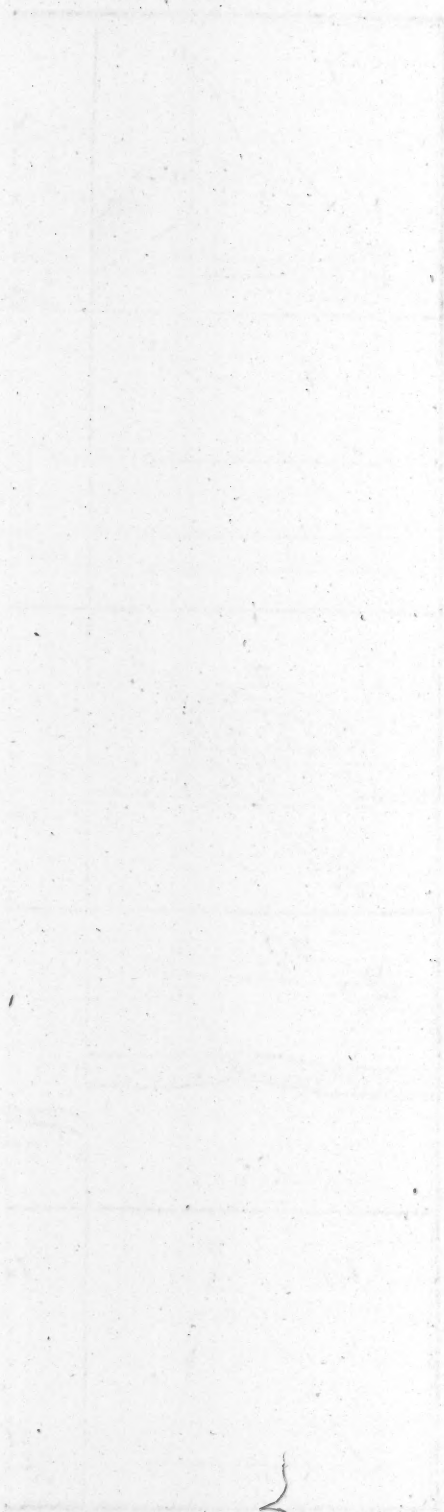




TRIGONOMETRY.

Plate IV.





Trigonometry

Plate V

